Hyperbolicity Cones of Elementary Symmetric Polynomials

Masterarbeit

vorgelegt von

Helena Bergold

an der

Universität Konstanz

Fachbereich Mathematik und Statistik

Betreuer: Prof. Dr. Markus Schweighofer

Konstanz, 2018

Erklärung der Selbständigkeit

Ich versichere hiermit, dass ich die vorliegende Arbeit mit dem Thema:

Hyperbolicity Cones of Elementary Symmetric Polynomials

selbstständig verfasst und keine anderen Hilfsmittel als die angegebenen benutzt habe. Die Stellen, die anderen Werken dem Wortlaut oder dem Sinne nach entnommen sind, habe ich in jedem einzelnen Falle durch Angaben der Quelle, auch der benutzten Sekundärliteratur, als Entlehnung kenntlich gemacht.

Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und auch noch nicht veröffentlicht.

Konstanz, den 16. Februar 2018

Helena Bergold

Contents

Introduction

We study the hyperbolicity cones of elementary symmetric polynomials and as the main result we show that these cones are spectrahedral. This claim was first conjectured by Sanyal [San11] and he showed that the $(n-1)$-th elementary symmetric polynomial in n variables has a spectrahedral hyperbolicity cone. In order to study the hyperbolicity cones, we need to introduce hyperbolic polynomials. This we do in the first chapter. The notion of hyperbolic polynomials goes back to the theory of partial differential equations (PDE) introduced by Petrovsky and Gårding [Brä13, p.1]. Besides the PDE theory, in the last years other mathematical fields such as combinatorics and convex optimisation increasingly showed interest in hyperbolic polynomials. The first one considering optimisation over hyperbolicity cones was Güler [Gül97].

In order to understand what the aim of studying hyperbolicity cones is, we need to have a closer look at optimisation. The best known case of optimisation is linear programming (LP). In this case we consider a linear function with linear equalities and inequalities as constraints. These constraint sets are polyhedrons. Since LP's are not sufficient for all optimisation problems, there is a generalisation, the semi-definite programming (SDP). In SDP's the constraint set is a spectrahedron. Every polyhedron is a spectrahedron. Still SDP does not cover all convex optimisation problems, so a further generalisation of SDP is the hyperbolic programming. The area considered in a hyperbolic program is a hyperbolicity cone and they are a generalisation of spectrahedrons.

Another question that might arise when regarding the hyperbolicity cones is how big the set of hyperbolicity cones is. Peter Lax conjectured in 1958 that all hyperbolicity cones of polynomials in at most three variables are spectrahedral [LPR03, p.1]. This statement stayed unproven for more than 40 years but was shown a few years ago [see LPR03; HV07]. However, it remains the open question whether all hyperbolicity cones are spectrahedral. This question is known as the Generalised Lax Conjecture [Brä13, p.2, Conjecture 1.1]. Mathematicians are still trying to prove the generalised conjecture. But until now it has remained unproven. There are not a lot of indications for the conjecture to be true though. Only some special cases have been shown. Beside the case of polynomials in at most three variables (Lax-Conjecture), the conjecture is true for quadratic polynomials (see [NS12]). In 2012, Brändén, a mathematician from Stockholm, showed that the hyperbolicity cones of elementary symmetric polynomials are spectrahedral. In order to show this statement, he used an important theorem from graph theory, the Matrix-Tree Theorem, which goes back to Kirchhoff and Maxwell [Brä13, p.3].

The Matrix-Tree Theorem shows that the spanning tree polynomial of a connected graph has a linear determinantal representation. Hence the hyperbolicity cones of spanning tree polynomials belonging to a connected graph are spectrahedral. In the second chapter, we introduce the notions and terms used in graph theory such that we are able to prove the Matrix-Tree Theorem in chapter three. In the last chapter, we recursively construct a graph $G_{n,k}$ for $n \geq k \geq 0$ such that the corresponding spanning tree polynomial has an elementary symmetric polynomial as a factor. This will lead to the main result: All hyperbolicity cones of elementary symmetric polynomials are spectrahedral. For the proof, we follow the idea of Brändén presented in [Brä13].

Chapter 1

Hyperbolic Polynomials and their Cones

In this thesis, we consider hyperbolicity cones of elementary symmetric polynomials. So in a first step, we need to define the essential terms belonging to the theory of hyperbolicity cones. This is what we want to do in this chapter. First, we study the hyperbolic polynomials and we will outline some of its properties. The following section is about the cones of hyperbolic polynomials, called the hyperbolicity cones. In the last section of this chapter, we are going to verify that certain directional derivatives of a hyperbolic polynomial is hyperbolic again. The most important result of this chapter is that all elementary symmetric polynomials are hyperbolic (see Proposition 1.3.9).

The origin of hyperbolic polynomials is the theory of partial differential equations, introduced by Petrovsky and Gårding (see [Brä13, p.1]). But in the last years, there is more and more interest in the hyperbolic polynomials in other areas of mathematics such as combinatorics and convex optimization [Brä13, p.1]. Güler, Lewis and Sendov developed the hyperbolic theory for convex analysis [Ren06, p.1].

1.1 Hyperbolic polynomials

As already mentioned, the definition of hyperbolic polynomials comes from the theory of partial differential equations. We are going to study hyperbolic polynomials with real coefficients but it is also possible to do this more generally in a finite dimensional euclidean space, for more details see [Ren06].

In a first step, we will introduce some notations used in this thesis.

1.1.1 Remark. (a) The natural numbers $\mathbb{N}$ are the positive integers, hence they do not contain the 0. For the non-negative integers, we write $\mathbb{N}_0$.

(b) We will use the notation $[n] := \{1, \ldots, n\}$ for any positive integer $n \in \mathbb{N}$.

(c) For this chapter, we fix an $n \in \mathbb{N}$ which denotes the number of variables. For any commutative ring R and any vector $\mathbf{x} \in R^n$, we write the vector $\mathbf{x}$ as an n-tuple of the form $\mathbf{x} = (x_1, \ldots, x_n)$. For our n variables $X_1, \ldots, X_n$, $\mathbf{X}$ is a notion for the n-tuple $\mathbf{X} = (X_1, \ldots, X_n)$. As another shortcut, we introduce $\mathbb{R}[\mathbf{X}] := \mathbb{R}[X_1, \ldots, X_n]$.

(d) Furthermore, we use the multi-index notation. An n-dimensional multi-index is an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ of non-negative integers with component-wise multiplication and addition. The absolute value of a multi-index $\alpha \in \mathbb{N}_0^n$ is

$$|\alpha| := \sum_{k=1}^{n} \alpha_k \in \mathbb{N}_0.$$

For any commutative ring R, we define for an element $\mathbf{x} = (x_1, \dots, x_n) \in R^n$ the term $\mathbf{x}^\alpha$ through $\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

(e) By the term 'degree' of a polynomial $p \in \mathbb{R}[X_1, \dots, X_n]$, we always think of the total degree of this polynomial p.

1.1.2 Definition. A polynomial $p \in \mathbb{R}[\mathbf{X}] = \mathbb{R}[X_1, \dots, X_n]$ is called *homogeneous* if p is a $\mathbb{R}$-linear combination of monomials of the same degree.

1.1.3 Remark. We regard polynomials, the elements of a polynomial ring $R[\mathbf{X}]$ for any commutative ring R, as a finite R-linear combination of monomials in the n variables $X_1, \dots, X_n$ (not as functions in an analytic meaning).

For any ring-extension $R' \supseteq R$ and any point $\mathbf{x} \in (R')^n$, we consider the polynomial evaluation homomorphism

$$\Phi_{\mathbf{x}} : R[\mathbf{X}] \to R', \quad p = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \mathbf{X}^\alpha \mapsto p(\mathbf{x}) := \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \mathbf{x}^\alpha,$$

where $c_\alpha \in R$ for every $\alpha \in \mathbb{N}_0^n$ and only finitely many c_α do not vanish such that we get a finite sum. For more details see [Bos09, p.58, Satz 5]. Nevertheless, we need to use some continuity arguments in the following work. For this reason we consider the polynomial function

$$\bar{p}: \ R^n \to R', \ \mathbf{x} = (x_1, \dots, x_n) \mapsto \bar{p}(\mathbf{x}) := \Phi_{\mathbf{x}}(p) = p(\mathbf{x})$$

for a fixed $p \in \mathbb{R}[\mathbf{X}]$. Instead of $\bar{p}(\mathbf{x})$ we just write $p(\mathbf{x})$ and often we say p is continuous. This polynomial function $\bar{p}$ is continuous in $\mathbf{x}$ and we often just say that p is continuous [DR11, p.48, 7.4(iii)].

In this work, we are mainly interested in the case $R = \mathbb{R}$ with ring-extension $R' = \mathbb{R}[T]$. In this case, the roots of the univariate polynomial $p(\mathbf{x} + T\mathbf{d})$, for a multivariate polynomial $p \in \mathbb{R}[\mathbf{X}]$ and two points $\mathbf{x}$ and $\mathbf{d}$ in $\mathbb{R}^n$ are continuous not only in the coefficients of the polynomial p but also in $\mathbf{x}$ and $\mathbf{d}$. This is because the coefficients are continuous in the points $\mathbf{x}, \mathbf{d}$. By this continuity, we mean:

1.1.4 Proposition. *[Bro13, p.23, Satz 16]. Let $f = \sum_{i=0}^{m} a_i T^i \in \mathbb{R}[T]$ be a polynomial of degree $m \in \mathbb{N}$ with roots $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ (counted with multiplicity). For any sequence $(f_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}[T]$ of polynomials of degree m converging coefficient-wise to f, i.e. if $f_k = \sum_{i=0}^{m} a_{i,k} T^k$ for all $k \in \mathbb{N}$, the coefficients $a_{i,k}$ converge to a_i for $k \to \infty$ and all $i \in \{0, \dots, m\}$. Then the roots $\alpha_{1,k}, \dots, \alpha_{m,k}$ (with multiplicity) of f_k converge to the roots of f, i.e. $\alpha_{i,k} \to \alpha_i$ for $k \to \infty$ and all $i \in \{0, \dots, m\}$ after rearranging the roots.*

Proof. We show the proposition by induction on the degree $m = \deg(f) \in \mathbb{N}$. For $m = 1$, it is

$$a_{1,k}(\alpha_1 - \alpha_{1,k}) = f_k(\alpha_1).$$

Since the coefficients $a_{0,k}$ and $a_{1,k}$ of f_k converge to the coefficients a_0 and a_1 of f, it follows $\lim_{k \to \infty} f_k(\alpha_1) = f(\alpha_1) = 0$. The leading coefficient a_1 of f does not vanish. This implies

$$(\alpha_1 - \alpha_{1,k}) \overset{k \to \infty}{\to} 0.$$

So clearly $\alpha_{1,k}$ converges to α_1 for $k \to \infty$.

For the induction step, we assume for a fixed $m > 1$ that for all polynomials $f \in \mathbb{R}[T]$ of degree $m-1$ and all sequences $(f_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}[T]$ with $\deg(f_k) = m-1$ converging coefficient-wise to f, the zeros $\alpha_{1,k}, \dots, \alpha_{m-1,k}$ of f_k converge to the zeros $\alpha_1, \dots, \alpha_{m-1}$ of f for $k \to \infty$. We want to show the statement for m. Again it is

$$a_{m,k}(\alpha_m - \alpha_{1,k}) \cdots (\alpha_m - \alpha_{m,k}) = f_k(\alpha_m) \overset{k\to\infty}{\to} f(\alpha_m) = 0.$$

Since $a_{m,k} \to a_m \neq 0$ for $k \to \infty$, we get

$$(\alpha_m - \alpha_{1,k}) \cdots (\alpha_m - \alpha_{m,k}) \overset{k\to\infty}{\to} 0.$$

So WLOG, we assume $\alpha_{m,k} \to \alpha_m$ for $k \to \infty$. It remains to show that the other roots converge as well. For this consider the polynomials

$$g := a_m \prod_{i=1}^{m-1} (z - \alpha_i) \quad \text{and} \quad g_k := a_{m,k} \prod_{i=1}^{m-1} (z - \alpha_{i,k}) \text{ for all } k \in \mathbb{N}.$$

Clearly, it is $f = (z - \alpha_m)g$ and $f_k = (z - \alpha_{m,k})g_k$ for all $k \in \mathbb{N}$. Let $g = \sum_{i=0}^{m-1} b_i T^i$ and $g_k = \sum_{i=0}^{m-1} b_{i,k} T^i$ denote the coefficients of g and g_k for all $k \in \mathbb{N}$. Then

$$\begin{aligned} a_m &= b_{m-1}, & a_i &= b_{i-1} - \alpha_m b_i \text{ for } i \in \{0, \dots, m-1\} \text{ and} \\ a_{m,k} &= b_{m-1,k}, & a_{i,k} &= b_{i-1,k} - \alpha_{m,k} b_{i,k} \text{ for } i \in \{0, \dots, m-1\},. \end{aligned}$$

It is easy to see that $b_{m-1,k} \to b_{m-1}$ for $k \to \infty$ for the other coefficients, it follows by induction. So $(g_k)_{k\in\mathbb{N}}$ converges coefficient-wise to g and $\deg(g) = \deg(g_k) = m-1$ for all $k \in \mathbb{N}$. The statement follows by the induction hypothesis. □

Now, we start with the theory of hyperbolic polynomials. So first, we define what is meant by this term.

1.1.5 Definition. [Brä13, p.1]. Let $p \in \mathbb{R}[\mathbf{X}] = \mathbb{R}[X_1, \dots, X_n]$ be a homogeneous polynomial of degree $m \in \mathbb{N}_0$ in the n variables $X_1, \dots, X_n$. We call p *hyperbolic in direction* $\mathbf{d} \in \mathbb{R}^n$, if for every $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial $p(\mathbf{x} + T\mathbf{d}) \in \mathbb{R}[T]$ has exactly m real roots counted with multiplicity.

A homogeneous polynomial $p \in \mathbb{R}[\mathbf{X}]$ is said to be *hyperbolic* if there exists a direction $\mathbf{d} \in \mathbb{R}^n$ such that p is hyperbolic in direction $\mathbf{d}$.

1.1.6 Remark. For arbitrary, fixed points $\mathbf{x}, \mathbf{d} \in \mathbb{R}^n$ and an arbitrary, fixed homogeneous polynomial $p \in \mathbb{R}[\mathbf{X}]$ of degree $m \in \mathbb{N}_0$, the polynomial $p(\mathbf{x} + T\mathbf{d}) \in \mathbb{R}[T]$ is a univariate polynomial of degree $m' \leq m$ (the zero polynomial is possible).

We can factorise it in the polynomial ring $\mathbb{C}[T]$ in such a way that all factors are linear, i.e.

$$p(\mathbf{x} + T\mathbf{d}) = c \prod_{k=1}^{m'} (T - r_k),$$

where $r_1, \dots, r_{m'}$ (with multiplicity) are the roots of $p(\mathbf{x} + T\mathbf{d})$ in $\mathbb{C}$ (not necessary real) and $c \in \mathbb{R}$ is the leading coefficient of $p(\mathbf{x} + T\mathbf{d})$. The zeros $r_1, \dots, r_{m'}$ and the coefficient depend on the direction $\mathbf{d}$ and the choice of the point $\mathbf{x}$. The dependency of the roots, we will study more in detail later on in this section (see Proposition 1.1.12).

Now, we want to determine the leading coefficient c more precisely. In order to do this, we assume $m' = m$ and write the homogeneous polynomial p as an $\mathbb{R}$-linear combination of monomials of degree m:

$$p = \sum_{\substack{\alpha \in \mathbb{N}_0^n, \\ |\alpha| = m}} c_\alpha \mathbf{X}^\alpha,$$

where all coefficients $c_\alpha \in \mathbb{R}$ are real. Evaluating our polynomial at $\mathbf{x} + T\mathbf{d} \in (\mathbb{R}[T])^n$ shows

$$p(\mathbf{x} + T\mathbf{d}) = \sum_{|\alpha| = m} c_\alpha (\mathbf{x} + T\mathbf{d})^\alpha.$$

This is a polynomial in $\mathbb{R}[T]$ of degree m with leading coefficient

$$c = \sum_{|\alpha| = m} c_\alpha \mathbf{d}^\alpha = p(\mathbf{d}).$$

Hence from now on we write the factorisation of $p(\mathbf{x} + T\mathbf{d})$ in the following form

$$p(\mathbf{x} + T\mathbf{d}) = p(\mathbf{d}) \prod_{k=1}^{m} (T - r_k).$$

If p is hyperbolic, all of the zeros mentioned above are real, and $m = m'$ is fulfilled (see proof of the next proposition), so we have the factorisation as above.

1.1.7 Proposition. *Let $p \in \mathbb{R}[\mathbf{X}]$ be a homogeneous polynomial with $\deg p = m \in \mathbb{N}_0$ and $\mathbf{d} \in \mathbb{R}^n$ any direction. The following characterisations are equivalent:*

(i) p is hyperbolic in direction $\mathbf{d}$

(ii) $p(\mathbf{d}) \neq 0$ and for every $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial $p(\mathbf{x} + T\mathbf{d})$ has only real roots

(iii) $p(\mathbf{d}) \neq 0$ and for every $\mathbf{x} \in \mathbb{R}^n$ there are m real roots $r_1, \dots, r_m$ (with multiplicity) of $p(\mathbf{x} + T\mathbf{d})$ in the factorisation $p(\mathbf{x} + T\mathbf{d}) = p(\mathbf{d}) \prod_{k=1}^{m} (T - r_k)$.

Proof. "$(i) \Rightarrow (ii)$": Let p be hyperbolic in direction $\mathbf{d}$. Since $p(\mathbf{x} + T\mathbf{d})$ has exactly m real roots for all $\mathbf{x} \in \mathbb{R}^n$, it is not possible that $p(\mathbf{d}) = 0$. If $p(\mathbf{d})$ was zero, $p(T\mathbf{d}) = T^m p(\mathbf{d})$ would be the zero polynomial. Hence for $\mathbf{x} = (0, \dots, 0) \in \mathbb{R}^n$, $p(\mathbf{x} + T\mathbf{d}) = p(T\mathbf{d})$ would have infinitely many roots. This is a contradiction, such that we get $p(\mathbf{d}) \neq 0$.

For every $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial $p(\mathbf{x} + T\mathbf{d})$ has degree m (see Remark 1.1.6). As a univariate polynomial of degree m, $p(\mathbf{x} + T\mathbf{d})$ has at most m different roots (in $\mathbb{C}$). By the assumption (i), there are exactly m real ones, which means $p(\mathbf{x} + T\mathbf{d})$ has only real roots.

"$(ii) \Rightarrow (i)$": Since $p(\mathbf{d}) \neq 0$, the leading coefficient does not vanish (1.1.6). Hence the univariate polynomial $p(\mathbf{x} + T\mathbf{d}) = p(\mathbf{d}) \prod_{k=1}^{m} (T - r_k)$ cannot be the zero-polynomial in $\mathbb{R}[T]$. Therefore $p(\mathbf{x} + T\mathbf{d})$ is a polynomial of degree m with exactly m roots in $\mathbb{C}$. All roots are real by assumption (ii), so we have exactly m real roots.

The equivalence "$(ii) \Leftrightarrow (iii)$" is trivial. □

1.1.8 Remark. It is also possible to define in a more general way whether a polynomial is hyperbolic in any direction $\mathbf{d} \in \mathbb{R}^n$. For example it is possible to define for an arbitrary (not necessary homogeneous) polynomial $p \in \mathbb{C}[X_1, \dots, X_n]$ if it is hyperbolic. For more details have a look at [Gül97, Definition 2.1].

That we only consider polynomials with real coefficients is up to the fact that for any hyperbolic polynomial $p \in \mathbb{C}[X_1, \dots X_n]$ (defined analogously as in Definition 1.1.5 with $\mathbb{C}[X_1, \dots X_n]$ instead of $\mathbb{R}[X_1, \dots, X_n]$) the polynomial $\frac{p}{p(\mathbf{d})}$ is a polynomial with real coefficients, since Proposition 1.1.7 holds equally and all roots r_k are real (look at the factorisation in 1.1.7 (iii)).

That we only consider homogeneous polynomials is because we are mainly interested in the hyperbolicity cones (introduced in the next section 1.2) and they depend only on the homogeneous part of highest degree of the polynomial. For more details considering this more general definition, see [Gül97, Definition 2.2].

1.1.9 Example. [Går59, p.3, Ex.1-4].

(1) One important example of a hyperbolic polynomial is $p_1 := \prod_{k=1}^{n} X_k \in \mathbb{R}[\mathbf{X}]$ which we are going to use later on. It is homogeneous of degree $m = n$ and it is hyperbolic in direction $\mathbf{d} = (1, \dots, 1) \in \mathbb{R}^n$, because $p_1(\mathbf{d}) = 1 \neq 0$ and for every $\mathbf{x} \in \mathbb{R}^n$ the zeros of the univariate polynomial

$$p_1(\mathbf{x} + T\mathbf{d}) = \prod_{k=1}^{n} (T + x_k)$$

are exactly all $-x_1, \dots, -x_m$. Since $\mathbf{x}$ was chosen as a real vector, all zeros are real. Hence the polynomial is hyperbolic in direction $\mathbf{d} = (1, \dots, 1)$ (see Proposition 1.1.7 (ii)).

Moreover, the polynomial p_1 is hyperbolic in any direction $\mathbf{d} \in \mathbb{R}^n$ with $p_1(\mathbf{d}) \neq 0$. To see this we use again (ii) of Proposition 1.1.7. For $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial

$$p_1(\mathbf{x} + T\mathbf{d}) = \prod_{k=1}^{n} (x_k + T d_k)$$

has the roots $-\frac{x_k}{d_k}$ for every $k \in [n]$ which are well-defined since $p_1(\mathbf{d}) \neq 0$ and therefore all entries of the vector $\mathbf{d}$ do not vanish. Furthermore, the roots $-\frac{x_k}{d_k}$ are real because $\mathbf{x}$ and $\mathbf{d}$ are real vectors.

(2) The polynomial $p_2 := X_1^2 - \sum_{k=2}^{n} X_k^2$ is hyperbolic in direction $\mathbf{d} = (1, 0, \dots, 0)$. In this case, we have a homogeneous polynomial of degree $m = 2$. Obviously $p_2(\mathbf{d}) = 1 \neq 0$ and

$$p_2(\mathbf{x} + T\mathbf{d}) = (x_1 + T)^2 - \sum_{k=2}^{n} x_k^2$$

has the two roots

$$t_1 = -x_1 + \sqrt{\sum_{k=2}^{n} x_k^2} \in \mathbb{R} \quad \text{and} \quad t_2 = -x_1 - \sqrt{\sum_{k=2}^{n} x_k^2} \in \mathbb{R}.$$

Since both of them are real, p_2 is hyperbolic in direction $\mathbf{d}$.

(3) Another important example is the determinant of symmetric matrices. A symmetric $k \times k$-matrix is determined by the upper triangular matrix, which consists of $n := \frac{k(k+1)}{2}$ entries. Let

$$\phi : \operatorname{Sym}_k(\mathbb{R}[\mathbf{X}]) \to (\mathbb{R}[\mathbf{X}])^n = (\mathbb{R}[\mathbf{X}])^{\frac{k(k+1)}{2}}$$

be an isomorphism between the symmetric $k \times k$ matrices and the vector space $(\mathbb{R}[\mathbf{X}])^n$. We consider the determinant of a symmetric matrix as a polynomial in those $n = \frac{k(k+1)}{2}$ entries, which are our n variables $X_1, \ldots, X_n$. For the n-tuple $\mathbf{X} = (X_1, \ldots, X_n)$, we define $X := \phi^{-1}(\mathbf{X}) \in \operatorname{Sym}_k(\mathbb{R}[\mathbf{X}])$ as the corresponding symmetric matrix. The determinant polynomial $p_3 := \det(\phi^{-1}(\mathbf{X})) = \det X \in \mathbb{R}[\mathbf{X}]$ is hyperbolic in direction $\mathbf{d} = \phi(I_k)$, where I_k is the $k \times k$ unit matrix. The reason for the hyperbolicity is that the zeros of the polynomial

$$p_3(\mathbf{x} + T\mathbf{d}) = \det(\phi^{-1}(\mathbf{x}) + T\phi^{-1}(\mathbf{d})) = \det(\phi^{-1}(\mathbf{x}) + TI_k)$$

for any $\mathbf{x} \in \mathbb{R}^n$ are up to sign the eigenvalues of the symmetric matrix $\phi^{-1}(\mathbf{x})$, which are real because of the symmetry.

The determinant-polynomial has degree $m = k$. To verify this have a look at the Leibniz-formula for determinants.

(4) An easy example of a hyperbolic polynomial is a constant polynomial $p_4 = a \in \mathbb{R}^\times$. This polynomial has degree $m = 0$ and no real roots but $p(\mathbf{d}) \neq 0$ for every $\mathbf{d} \in \mathbb{R}^n$.

As we have seen in example (3), for the determinant polynomial p_3 the roots of $p_3(\mathbf{x}+T\mathbf{d})$ for any vector $\mathbf{x} \in \mathbb{R}^n$ are minus the eigenvalues of the corresponding matrix $\phi^{-1}(\mathbf{x})$. From linear algebra the term characteristic polynomial for a matrix $A \in \mathrm{M}_k(\mathbb{R})$ is known as the polynomial $P_A = \det(TI_k - A) \in \mathbb{R}[T]$ and the roots of this polynomial are the eigenvalues of A. We want to generalise this terminology to hyperbolic polynomials in the following definition.

1.1.10 Definition. Let p be hyperbolic in direction $\mathbf{d} \in \mathbb{R}^n$ with $\deg(p) = m$. Let $\mathbf{x}$ be an arbitrary point in $\mathbb{R}^n$. The *characteristic polynomial* of $\mathbf{x}$ with respect to p in direction $\mathbf{d}$ is said to be $p(T\mathbf{d} - \mathbf{x})$ and the roots of the characteristic polynomial $p(T\mathbf{d} - \mathbf{x})$ are called the *eigenvalues* of $\mathbf{x}$ with respect to p in direction $\mathbf{d}$. There are m of those roots counted with multiplicity for every direction $\mathbf{d} \in \mathbb{R}^n$ in which p is hyperbolic and every point $\mathbf{x} \in \mathbb{R}^n$, denoted by $\lambda_1(\mathbf{d}, \mathbf{x}), \ldots, \lambda_m(\mathbf{d}, \mathbf{x})$.

Since $p(T\mathbf{d}-\mathbf{x}) = p((-\mathbf{x})+T\mathbf{d})$ has only real roots for a hyperbolic polynomial p, all eigenvalues $\lambda_1(\mathbf{d}, \mathbf{x}), \ldots, \lambda_m(\mathbf{d}, \mathbf{x})$ are real.

1.1.11 Proposition. *Let $p \in \mathbb{R}[\mathbf{X}]$ be a polynomial, hyperbolic in direction $\mathbf{d}$, and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. The eigenvalues of $\mathbf{x}$ with respect to p in direction $\mathbf{d}$ are minus the roots of $p(\mathbf{x} + T\mathbf{d})$.*

Proof. Similar to 1.1.6 one can show that

$$p(T\mathbf{d} - \mathbf{x}) = p(\mathbf{d}) \prod_{k=1}^{m} (T - \lambda_k(\mathbf{d}, \mathbf{x})).$$

On the other hand

$$p(T\mathbf{d} - \mathbf{x}) = (-1)^m p(\mathbf{x} + (-T)\mathbf{d}) = (-1)^m p(\mathbf{d}) \prod_{k=1}^{m} (-T - r_k) = p(\mathbf{d}) \prod_{k=1}^{m} (T + r_k).$$

Hence the eigenvalues $\lambda_k(\mathbf{d}, \mathbf{x}) = -r_l$ are minus the zeros of $p(\mathbf{x} + T\mathbf{d})$ for any $l, k \in [m]$. $\square$

So from now on, we consider the eigenvalues instead of the roots of hyperbolic polynomials and always write the factorisation as

$$p(\mathbf{x} + T\mathbf{d}) = p(\mathbf{d}) \prod_{k=1}^{m} (T + \lambda_k(\mathbf{d}, \mathbf{x}))$$

for a polynomial p of degree m, which is hyperbolic in direction $\mathbf{d}$ and for every $\mathbf{x} \in \mathbb{R}^n$. Furthermore, evaluating $p(\mathbf{x} + T\mathbf{d})$ at the point 0 shows

$$p(\mathbf{x} + T\mathbf{d})|_{T=0} = p(\mathbf{x}) = p(\mathbf{d}) \prod_{k=1}^{m} \lambda_k(\mathbf{d}, \mathbf{x}) \tag{1.1}$$

for every $\mathbf{x} \in \mathbb{R}^n$. The notation $f|_{T=0}$ for a polynomial f in the variable T means that we evaluate the polynomial f at the point 0.

The eigenvalues have some special properties as a function of the direction $\mathbf{d}$ and the vector $\mathbf{x}$.

1.1.12 Proposition. *[Ren06, p.2] and [Går59, p.2]. The eigenvalues of a hyperbolic polynomial p of degree m for any direction $\mathbf{d} \in \mathbb{R}^n$ and any $\mathbf{x} \in \mathbb{R}^n$ as defined in the previous definition are real, so we can order them. We assume $\lambda_1(\mathbf{d}, \mathbf{x}) \leq \ldots \leq \lambda_m(\mathbf{d}, x)$. Furthermore, they fulfil the following equation*

$$\forall s, t \in \mathbb{R}: \ \lambda_k(\mathbf{d}, t\mathbf{x} + s\mathbf{d}) = \begin{cases} t\lambda_k(\mathbf{d}, \mathbf{x}) + s, & \text{if } t \geq 0; \\ t\lambda_{m-k+1}(\mathbf{d}, \mathbf{x}) + s, & \text{if } t < 0 \end{cases} \tag{1.2}$$

for every $k \in [m]$. If p is hyperbolic in direction $\mathbf{d}$, it is also hyperbolic in direction $t\mathbf{d}$ for any $t \in \mathbb{R}^{\times}$. More generally, the following connection between the eigenvalues in direction $\mathbf{d}$ and $t\mathbf{d}$ holds

$$\forall t \in \mathbb{R}^{\times}: \ \lambda_k(t\mathbf{d}, \mathbf{x}) = \begin{cases} \frac{1}{t}\lambda_k(\mathbf{d}, \mathbf{x}), & \text{if } t \geq 0; \\ \frac{1}{t}\lambda_{m-k+1}(\mathbf{d}, \mathbf{x}) & \text{if } t < 0 \end{cases} \tag{1.3}$$

for all $k \in [m]$.

Proof. As a first step, we show that the eigenvalues fulfil

$$\forall k \in [m] : \forall t \in \mathbb{R} : \lambda_k(\mathbf{d}, t\mathbf{x}) = \begin{cases} t \cdot \lambda_k(\mathbf{d}, \mathbf{x}), & \text{if } t \geq 0; \\ t \cdot \lambda_{m-k+1}(\mathbf{d}, \mathbf{x}), & \text{if } t < 0. \end{cases} \tag{1.4}$$

Therefore, we consider the factorisation of the polynomial $p(\mathbf{x}+T\mathbf{d})$. Let us first have a look at the case $t = 0$. In this case the right-hand side of the equation (1.4) is obviously zero for every $k \in [m]$ and for the polynomial $p(t\mathbf{x} + T\mathbf{d}) = p(\mathbf{0} + T\mathbf{d})$ we get:

$$p(\mathbf{0} + T\mathbf{d}) = T^m p(\mathbf{d}) = p(\mathbf{d}) \prod_{k=1}^{m} T.$$

This means that all eigenvalues $\lambda_k(\mathbf{d}, 0 \cdot \mathbf{x}) = \lambda_k(\mathbf{d}, \mathbf{0})$ are zero. Hence the equation (1.4) is fulfilled.

In a second case, we assume $t \neq 0$ to get

$$\begin{aligned} p(\mathbf{d}) \prod_{k=1}^{m} (T + \lambda_k(\mathbf{d}, t\mathbf{x})) &= p(t\mathbf{x} + T\mathbf{d}) \\ &\overset{t \neq 0}{=} p\left(t\left(\mathbf{x} + \frac{T}{t}\mathbf{d}\right)\right) \\ &= t^m \cdot p\left(\mathbf{x} + \frac{T}{t}\mathbf{d}\right) \\ &= t^m \cdot p(\mathbf{d}) \prod_{k=1}^{m} \left(\frac{T}{t} + \lambda_k(\mathbf{d}, \mathbf{x})\right) \\ &= p(\mathbf{d}) \prod_{k=1}^{m} (T + t\lambda_k(\mathbf{d}, \mathbf{x})). \end{aligned}$$

So we get $\lambda_k(\mathbf{d}, t\mathbf{x}) = t \cdot \lambda_l(\mathbf{d}, \mathbf{x})$ for some $k, l \in [m]$. Since we ordered the eigenvalues ascending such that $\lambda_1(\mathbf{d}, \mathbf{x}) \leq \lambda_2(\mathbf{d}, \mathbf{x}) \leq \ldots \leq \lambda_m(\mathbf{d}, \mathbf{x})$ and this inequalities are stable under multiplication with a real number $t > 0$ and get reversed by multiplication with a real number $t < 0$, we get

$$\lambda_k(\mathbf{d}, t\mathbf{x}) = \begin{cases} t \cdot \lambda_k(\mathbf{d}, \mathbf{x}), & \text{if } t \geq 0; \\ t \cdot \lambda_{m-k+1}(\mathbf{d}, \mathbf{x}), & \text{if } t < 0. \end{cases}$$

To show the first equation (1.2) of the proposition, assume s, t are arbitrary real numbers. As we have seen, we can factorise the polynomial in the following way

$$p((t\mathbf{x} + s\mathbf{d}) + T\mathbf{d}) = p(\mathbf{d}) \prod_{k=1}^{m} (T + \lambda_k(\mathbf{d}, t\mathbf{x} + s\mathbf{d})),$$

since $t\mathbf{x}+s\mathbf{d}$ is a vector in $\mathbb{R}^n$. On the other hand, it is possible to rewrite it as follows:

$$\begin{aligned} p\left((t\mathbf{x}+s\mathbf{d})+T\mathbf{d}\right) &= p\left(t\mathbf{x}+(T+s)\mathbf{d}\right) \\ &= p(\mathbf{d})\prod_{k=1}^{m}\left((T+s)+\lambda_k(\mathbf{d},t\mathbf{x})\right) \\ &\overset{(1.4)}{=} p(\mathbf{d})\prod_{k=1}^{m}\left(T+(s+t\lambda_k(\mathbf{d},\mathbf{x}))\right). \end{aligned}$$

In the last step, we used the homogeneity of the eigenvalues in the second argument (see equation (1.4)). Analogue to the previous part using the ordering of the eigenvalues, we get the first part of the claim. For the second statement (1.3), we look at the equation

$$\begin{aligned} p(t\mathbf{d})\prod_{k=1}^{m}\left(T+\lambda_k(t\mathbf{d},\mathbf{x})\right) &= p\left(\mathbf{x}+T(t\mathbf{d})\right) \\ &= p\left(\mathbf{x}+(Tt)\mathbf{d}\right) \\ &= p(\mathbf{d})\prod_{k=1}^{m}\left(Tt+\lambda_k(\mathbf{d},\mathbf{x})\right) \\ &= t^m\cdot p(\mathbf{d})\prod_{k=1}^{m}\left(T+\frac{1}{t}\lambda_k(\mathbf{d},\mathbf{x})\right) \\ &= p(t\mathbf{d})\prod_{k=1}^{m}\left(T+\frac{1}{t}\lambda_k(\mathbf{d},\mathbf{x})\right). \end{aligned}$$

By this equality, we get analogously to the previous part the claimed statement. □

1.2 Hyperbolicity cones

The theory of hyperbolicity cones is used for hyperbolic programs, which extends the theory of semi-definite programming (SDP). This, we are going to see in Example 1.2.9. The main result of this section is that all hyperbolicity cones are convex cones.

We already mentioned that the eigenvalues $\lambda_1(\mathbf{d},\mathbf{x}),\ldots,\lambda_m(\mathbf{d},\mathbf{x})$ of a hyperbolic polynomial p are continuous in $\mathbf{x}$ and in $\mathbf{d}$ each as a function from $\mathbb{R}^n$ to $\mathbb{R}$ (see 1.1.3 and 1.1.11). So we want to define a set in $\mathbb{R}^n$ which is a cone and in which all eigenvalues have the same sign. This set, we are going to call the hyperbolicity cone.

1.2.1 Definition. [Ren06, p.2]. Let p be a polynomial, hyperbolic in direction $\mathbf{d}\in\mathbb{R}^n$. The set

$$\Lambda(p,\mathbf{d}) := \{\mathbf{x}\in\mathbb{R}^n :\ \forall k\in[m] :\ \lambda_k(\mathbf{d},\mathbf{x})>0\}$$

is called the *open hyperbolicity cone of p in direction* $\mathbf{d}$. If for $\mathbf{x}\in\mathbb{R}^n$ the smallest eigenvalue of p is denoted by $\lambda_1(\mathbf{d},\mathbf{x})$ the open hyperbolicity cone is $\Lambda(p,\mathbf{d})=\{\mathbf{x}\in\mathbb{R}^n :\ \lambda_1(\mathbf{d},\mathbf{x})>0\}$.

1.2.2 Remark. The open hyperbolicity cone as defined above is an open set in $\mathbb{R}^n$. To show this, consider the eigenvalues for a fixed $\mathbf{d} \in \mathbb{R}^n$ as a function from $\mathbb{R}^n$ to $\mathbb{R}$ in the second argument. The hyperbolicity cone is then

$$\Lambda(p, \mathbf{d}) = \bigcap_{k=1}^{m} \lambda_k^{-1}(\mathbf{d}, (0, \infty)).$$

Since the eigenvalues are continuous (see 1.1.3 and 1.1.11) and the pre-image of an open set is open, the open hyperbolicity cone is open.

1.2.3 Remark. Since p is hyperbolic in direction $\mathbf{d}$, the point $\mathbf{d}$ itself is an element of the open hyperbolicity cone $\Lambda(p, \mathbf{d})$ of p in direction $\mathbf{d}$. Since p is homogeneous of degree m we get

$$p(\mathbf{d} + T\mathbf{d}) = p\left((1 + T)\mathbf{d}\right) = p(\mathbf{d})(T + 1)^m.$$

Hence $\lambda_k(\mathbf{d}, \mathbf{d}) = 1 > 0$ for every $k \in [m]$, especially for $k = 1$.

1.2.4 Proposition. *[Ren06, p.2]. For every in direction $\mathbf{d}$ hyperbolic polynomial p, the open hyperbolicity cone $\Lambda(p, \mathbf{d})$ is an open cone, i.e. it is closed under multiplication with positive scalars.*

Proof. Let us start with an element $\mathbf{x} \in \Lambda(p, \mathbf{d})$ of the open hyperbolicity cone. By application of Proposition 1.1.12, we get $\lambda_1(\mathbf{d}, t\mathbf{x}) = t\lambda_1(\mathbf{d}, \mathbf{x})$, which is positive for any $t > 0$ since $\lambda_1(\mathbf{d}, \mathbf{x}) > 0$ by the assumption $\mathbf{x} \in \Lambda(p, \mathbf{d})$. $\square$

In this section, we want to show that $\Lambda(p, \mathbf{d})$ is not only an open cone but also convex. Afterwards, we will study the closure of the open convex cone to work with it later on in chapter four. To prove the convexity of the open hyperbolicity cone, we first study different presentations of the cone. In order to do so, we use some continuity arguments.

1.2.5 Proposition. *[Ren06, Proposition 1]. The open hyperbolicity cone of a hyperbolic polynomial p in direction $\mathbf{d}$ is the connected component of $\{\mathbf{x} \in \mathbb{R}^n : \ p(\mathbf{x}) \neq 0\}$ containing $\mathbf{d}$.*

Proof. Let S denote the connected component of $\{\mathbf{x} \in \mathbb{R}^n : \ p(\mathbf{x}) \neq 0\}$ containing the point $\mathbf{d}$.

First, we want to show the part $S \subseteq \Lambda(p, \mathbf{d})$. Since $\Lambda(p, \mathbf{d})$ is open in $\mathbb{R}^n$ (1.2.2), the intersection $\Lambda(p, \mathbf{d}) \cap S$ is open in S. Furthermore, $\{\mathbf{x} \in \mathbb{R}^n : \exists k \in [m] : \ \lambda_k(\mathbf{d}, \mathbf{x}) < 0\} = \bigcup_{k=1}^{m} \lambda_k(\mathbf{d}, (-\infty, 0))$ is open, too. The set S satisfies

$$S = (\{\mathbf{x} \in \mathbb{R}^n : \ \exists k \in [m] : \ \lambda_k(\mathbf{d}, \mathbf{x}) < 0\} \cap S) \,\dot{\cup}\, (\Lambda(p, \mathbf{d}) \cap S)$$

But S is connected, so one of the both unified sets must be the empty-set. By the definition of S, we know $\mathbf{d} \in S$ and $\mathbf{d} \in \Lambda(p, \mathbf{d})$ (see 1.2.3) implies that $\Lambda(p, \mathbf{d}) \cap S = S$. This shows $S \subseteq \Lambda(p, \mathbf{d})$.

To show the equality of the two cones, we only need to prove that $\Lambda(p, \mathbf{d})$ is connected because $\Lambda(p, \mathbf{d}) \subseteq \{\mathbf{x} \in \mathbb{R}^n : \ p(\mathbf{x}) \neq 0\}$. To show the connectivity, it is sufficient to prove that for an arbitrary $\mathbf{x} \in \mathbb{R}^n$ there is always a path from $\mathbf{x}$ to $\mathbf{d}$ in $\Lambda(p, \mathbf{d})$. We are going to show that the line segment $l := \{t\mathbf{d} + (1-t)\mathbf{x} : \ t \in [0,1]\}$ is completely contained in the open cone $\Lambda(p,\mathbf{d})$. This follows from the properties of the eigenvalues (see Proposition 1.1.12):

$$\lambda_k(\mathbf{d}, t\mathbf{d} + (1-t)\mathbf{x}) = t \cdot \underbrace{\lambda_k(\mathbf{d},\mathbf{x})}_{>0, \text{ since } \mathbf{x}\in\Lambda(p,\mathbf{d})} + (1-t) > 0,$$

for all $t \in [0,1]$ and $k \in [m]$. Hence $l \subseteq \Lambda(p,\mathbf{d})$ and the open hyperbolicity cone $\Lambda(p,\mathbf{d})$ is connected. □

1.2.6 Remark. We showed that the hyperbolicity cone of an in direction $\mathbf{d}$ hyperbolic polynomial p is star shaped with respect to $\mathbf{d}$.

1.2.7 Corollary. *Let p be a hyperbolic polynomial with respect to $\mathbf{d}$ and $\mathbf{x} \in \Lambda(p,\mathbf{d})$. Then the line $l := \{t\mathbf{d} + (1-t)\mathbf{x} : \ t \in [0,1]\}$ is contained in the hyperbolicity cone $\Lambda(p,\mathbf{d})$.*

Proof. See the last part of the previous proof. □

Now, we introduced the elementary definitions for the theory of hyperbolicity cones. The aim of this section, is to show that all hyperbolicity cones are convex. Furthermore, we want to study the hyperbolicity cones of elementary symmetric polynomials in this thesis. The overall aim is to show that these cones are spectrahedral. For this, we need to define what a spectrahedral cone is.

1.2.8 Definition. A spectrahedral cone in $\mathbb{R}^n$ is a cone of the form

$$\left\{\mathbf{x} \in \mathbb{R}^n : \ \sum_{i=1}^{n} x_i A_i \succeq 0\right\}$$

for symmetric matrices $A_1, \ldots, A_n \in \mathrm{Sym}_k(\mathbb{R})$ for a $k \in \mathbb{N}$ such that there exists a vector $\mathbf{y} \in \mathbb{R}^n$ with $\sum\limits_{i=1}^{n} y_i A_i \succ 0$.

The existence of the vector $\mathbf{y}$ with $\sum\limits_{i=1}^{n} y_i A_i \succ 0$ ensures that the interior of the cone is non-empty.

Let us now consider some examples of hyperbolicity cones. For this, we use the hyperbolic polynomials mentioned in Example 1.1.9 and study their cones.

1.2.9 Example.

(1) For the polynomial $p_1 = \prod\limits_{k=1}^{n} X_k$ the hyperbolicity cone $\Lambda(p_1,\mathbf{d})$ is the positive orthant if and only if all entries of $\mathbf{d}$ are positive, for instance if $\mathbf{d} = (1,\ldots,1) \in \mathbb{R}^n$. For an arbitrary $\mathbf{d} \in (\mathbb{R}^\times)^n$ the hyperbolicity cone of p_1 is the orthant in which the direction $\mathbf{d}$ is included. In the case $n = 2$, there are four quadrants and the same number of possible hyperbolicity

cones depending on the direction $\mathbf{d}$.

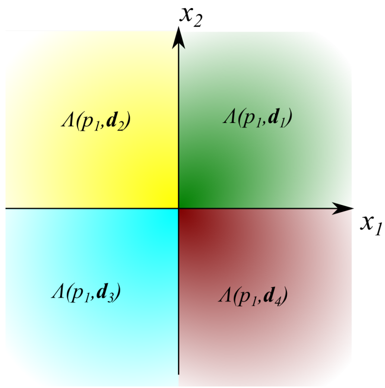


Figure 1.1: Hyperbolicity cones of $p_1 = X_1X_2$ depending on the direction $\mathbf{d}$. If $\mathbf{d}_1 \in \mathbb{R}^2_+$ the hyperbolicity cone is the first quadrant (green), $\mathbf{d}_2 \in \mathbb{R}_- \times \mathbb{R}_+$: second quadrant (yellow), $\mathbf{d}_3 \in \mathbb{R}^2_-$ the third quadrant (blue) and $\mathbf{d}_4 \in \mathbb{R}_+ \times \mathbb{R}_-$ fourth quadrant (red). With $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_- := \{x \in \mathbb{R} : x < 0\}$.

In this special case of hyperbolic polynomials, we are in the case of linear programming (LP) since the hyperbolicity cone is a polyhedron, which is the type of cone we need as constraint set for a LP. As a reminder, a cone is called *polyhedron* if and only if there is a presentation as an intersection of finitely many half-spaces.

More generally, a homogeneous polynomial $p = \prod_{k=1}^{m} l_k$ which consists only out of linear factors $l_1, \ldots, l_m \in \mathbb{R}[\mathbf{X}]$ is hyperbolic and its hyperbolicity cones is a polyhedra. The reason for p being hyperbolic is that for all $\mathbf{x} \in \mathbb{R}^n$ and any direction $\mathbf{d} \in \mathbb{R}^n$ with $l_k(\mathbf{d}) \neq 0$ for all $k \in [m]$:

$$p(\mathbf{x} + T\mathbf{d}) = \prod_{k=1}^{m} l_k(\mathbf{x} + T\mathbf{d}) = \prod_{k=1}^{m} \left(l_k(\mathbf{x}) + Tl_k(\mathbf{d})\right).$$

The zeros of this univariate polynomial are $-\frac{l_k(\mathbf{x})}{l_k(\mathbf{d})}$ which are real numbers because $\mathbf{x}, \mathbf{d} \in \mathbb{R}^n$ and $l_k \in \mathbb{R}[\mathbf{X}]$ for all $k \in [m]$.

(2) The hyperbolicity cone of the polynomial $p_2 = X_1^2 - \sum_{k=2}^{n} X_k^2$ is the forward light cone. For $n = 2$, we get the following cone:

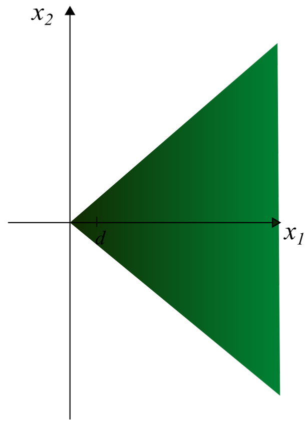


Figure 1.2: The hyperbolicity cone $\Lambda(p_2, \mathbf{d})$ for $p_2 = X_1^2 - X_2^2$ and $\mathbf{d} = (1, 0)$ in two dimensions.

For $n = 3$ it is:

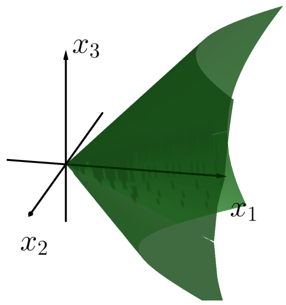


Figure 1.3: Hyperbolicity cone $\Lambda(p_2, \mathbf{d})$ in three dimensions with $\mathbf{d} = (1, 0, 0)$.

(3) Now, we consider again $p_3 = \det X$ for any matrix $X = \phi^{-1}(\mathbf{X})$ and study the corresponding hyperbolicity cone of this polynomial in direction $\mathbf{d} = \phi^{-1}(I_k)$. For the notation used here, see 1.1.9 (3). It is defined as

$$\Lambda(p_3, \mathbf{d}) = \{\mathbf{x} \in \mathbb{R}^n : \forall k \in [m] : \lambda_k(\mathbf{d}, \mathbf{x}) > 0\}$$

and the eigenvalues of p_3 (see Definition 1.1.10) are exactly the eigenvalues of the matrix X. So this cone is the set of positive definite matrices. The closure is then the cone of positive semi-definite matrices which is spectrahedral. It is clear that every symmetric matrix whose entries are either homogeneous polynomials of degree one or vanish has a presentation $\sum_{i=1}^{n} X_i A_i$ with symmetric matrices $A_1, \ldots, A_n$ and so the hyperbolicity cone of the determinant of this matrix is spectrahedral. Since all spectrahedral cones are determined by such a matrix polynomial, every spectrahedral cone is a hyperbolicity cone. This we are going to show in the following Proposition 1.2.11. It is natural to ask whether the other inclusion holds as well. 1958 Lax conjectured that all hyperbolicity cones of polynomials in maximum three variables are spectrahedral. This conjecture is already proven see

[LPR03] and [HV07]. The generalized Lax-Conjecture says that all hyperbolicity cones are spectrahedral. Beside the work of Lewis, Parrilo and Ramana, it is also true for quadratic polynomials [NS12]. But in general, there are only a few evidence in favour of this conjecture. Zinchenko showed in [Zin08] that all hyperbolicity cones of elementary symmetric polynomials are spectrahedral shadows. Brändén showed that those hyperbolicity cones are already spectrahedral cones [Brä13]. This proof of Brändén is the aim of this thesis.

(4) The hyperbolicity cone of a constant polynomial $p_4 = a \in \mathbb{R}^\times$ is

$$\Lambda(p_4, \mathbf{d}) = \mathbb{R}^n.$$

Let us now define the closure of the open hyperbolicity cone.

1.2.10 Definition. The closure $\overline{\Lambda}(p, \mathbf{d}) := \overline{\Lambda(p, \mathbf{d})}$ of the open hyperbolicity cone $\Lambda(p, \mathbf{d})$ is said to be the *(closed) hyperbolicity cone* of p in direction $\mathbf{d}$. If we just say, hyperbolicity cone of p in direction $\mathbf{d}$, we always speak of the closed hyperbolicity cone.

1.2.11 Proposition. *[LPR03, Proposition 2]. All spectrahedral cones are (closed) hyperbolicity cones.*

Proof. Let

$$S = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i A_i \succeq 0 \right\}$$

be a spectrahedral cone with symmetric matrices $A_1, \ldots, A_n \in \mathrm{Sym}_k(\mathbb{R})$ for any $k \in \mathbb{N}$ and such that there is a $\mathbf{y} \in \mathbb{R}^n$ with $\sum_{i=1}^n y_i A_i \succ 0$. We claim that the polynomial $p := \det\left(\sum_{i=1}^n X_i A_i\right) \in \mathbb{R}[\mathbf{X}]$ is hyperbolic in direction $\mathbf{y}$.

First note that $p(\mathbf{y}) > 0$. Let $A := \sum_{i=1}^n y_i A_i$ be the symmetric, positive definite matrix and $A^{1/2}$ its square root. The matrix $A^{1/2}$ is also symmetric and positive definite. For any vector $\mathbf{x} \in \mathbb{R}^n$, we need to show that $p(\mathbf{x} + T\mathbf{y})$ has only real zeros.

$$\begin{aligned}
p(\mathbf{x} + T\mathbf{y}) &= \det\left(\sum_{i=1}^n (x_i + Ty_i) A_i\right) \\
&= \det\left(A^{1/2} A^{-1/2} \left(\sum_{i=1}^n x_i A_i\right) A^{-1/2} A^{1/2} + TA\right) \\
&= \det(A^{1/2}) \det\left(A^{-1/2} \left(\sum_{i=1}^n x_i A_i\right) A^{-1/2} + TI_k\right) \det(A^{1/2}) \\
&= \underbrace{\det(A)}_{>0} \det\left(A^{-1/2} \left(\sum_{i=1}^n x_i A_i\right) A^{-1/2} + TI_k\right).
\end{aligned}$$

This polynomial has only real roots because the matrix $A^{-1/2}\left(\sum_{i=1}^n x_i A_i\right) A^{-1/2}$ is symmetric and therefore it has only real eigenvalues. So p is hyperbolic in direction $\mathbf{y}$.

Furthermore, the eigenvalues of p coincide with the eigenvalues of the matrix

$$A^{-1/2}\left(\sum_{i=1}^{n} x_i A_i\right) A^{-1/2}$$

and these eigenvalues are positive if and only if the eigenvalues of $\sum\limits_{i=1}^{n} x_i A_i$ are positive because $A^{1/2}$ is positive definite. This shows the equality of the cones. $\square$

1.2.12 Proposition. *Let p be a polynomial in $\mathbb{R}[\mathbf{X}]$ of degree m, which is hyperbolic in direction $\mathbf{d} \in \mathbb{R}^n$. There are different presentations of the open hyperbolicity cone.*

(i) $\Lambda(p, \mathbf{d}) = \{\mathbf{x} \in \mathbb{R}^n : \ \forall k \in [m] : \lambda_k(\mathbf{d}, \mathbf{x}) > 0\}$.

(ii) $\Lambda(p, \mathbf{d})$ *is the connected component of* $\{\mathbf{x} \in \mathbb{R}^n : \ p(\mathbf{x}) \neq 0\}$ *containing* $\mathbf{d}$ *itself.*

(iii) $\Lambda(p, \mathbf{d}) = \{\mathbf{x} \in \mathbb{R}^n : \ \forall t \geq 0 : \ p(\mathbf{x} + t\mathbf{d}) \neq 0\}$.

Proof. (i) is by definition of the hyperbolicity cones 1.2.1, (ii) holds by Proposition 1.2.5.
The third presentation follows directly from the fact that all roots of the polynomial $p(\mathbf{x}+T\mathbf{d})$ are (-1) times the eigenvalues. $\square$

In the next part, we figure out some properties of the open hyperbolicity cones.

1.2.13 Proposition. *[Går59, p.4]. Let $p \in \mathbb{R}[\mathbf{X}]$ be polynomial of degree $[m]$, hyperbolic in direction $\mathbf{d} \in \mathbb{R}^n$.*

(i) $\Lambda(p, -\mathbf{d}) = -\Lambda(p, \mathbf{d})$.

(ii) $\Lambda(p, \mathbf{d}) = t\Lambda(p, \mathbf{d}) = \Lambda(p, t\mathbf{d})$ *for any* $t > 0$.

Proof. Let us first proof the first equation (i) for the open hyperbolicity cones. We start with an element $\mathbf{x} \in \Lambda(p, \mathbf{d})$ and need to show that $-\mathbf{x} \in \Lambda(p, -\mathbf{d})$. With other words, For all $k \in [m]$ we know $\lambda_k(\mathbf{d}, \mathbf{x}) > 0$ and want to prove that then $\lambda_k(-\mathbf{d}, -\mathbf{x})$ is positive, too. For this, we just need to use the properties of the eigenvalues shown in Proposition 1.1.12. For $k \in [m]$

$$\lambda_k(-\mathbf{d}, -\mathbf{x}) \overset{(1.3)}{=} -\lambda_{m-k+1}(\mathbf{d}, -\mathbf{x}) \overset{(1.2)}{=} \lambda_k(\mathbf{d}, \mathbf{x}) > 0.$$

This shows the first inclusion. For the other inclusion, let $\mathbf{x} \in \Lambda(p, -\mathbf{d})$, i.e $\lambda_k(-\mathbf{d}, \mathbf{x}) > 0$ for all $k \in [m]$ and again with the properties of the eigenvalues, we get

$$\lambda_k(\mathbf{d}, -\mathbf{x}) = \lambda_k(-\mathbf{d}, \mathbf{x}) > 0.$$

Hence $-\mathbf{x}$ is in the open hyperbolicity cone $\Lambda(p, \mathbf{d})$.
The second statement follows directly from the fact that the open hyperbolicity cones are closed under multiplication with a positive number (see Proposition 1.2.4) and the property $\frac{1}{t}\lambda_1(\mathbf{d}, \mathbf{x}) = \lambda_1(t\mathbf{d}, \mathbf{x})$ for any $t > 0$ of the eigenvalues, shown in Proposition 1.1.12. With other words, multiplication with a positive real number $t > 0$ does not change anything with the sign of the eigenvalues. $\square$

1.2.14 Theorem. *[Ren06, Theorem 3]. Let p be hyperbolic in direction $\mathbf{d}$. If $\mathbf{e} \in \Lambda(p, \mathbf{d})$, then p is hyperbolic in direction $\mathbf{e}$. Moreover $\Lambda(p, \mathbf{d}) = \Lambda(p, \mathbf{e})$.*

Proof. Let $\mathbf{e}$ be a point of the open cone $\Lambda(p, \mathbf{d})$. We want to show that p is hyperbolic in direction $\mathbf{e}$, which means by definition that the univariate polynomial $p(\mathbf{x} + T\mathbf{e})$ has only real roots for all $\mathbf{x} \in \mathbb{R}^n$.From now on, fix an arbitrary point $\mathbf{x} \in \mathbb{R}^n$.

By the assumption $\mathbf{e} \in \Lambda(p, \mathbf{d})$, we get $p(\mathbf{e}) = p(\mathbf{d}) \prod_{k=1}^{m} \lambda_k(\mathbf{d}, \mathbf{e}) \neq 0$ (see (1.1)) and $\operatorname{sgn}(p(\mathbf{e})) = \operatorname{sgn}(p(\mathbf{d}))$. WLOG, we assume $p(\mathbf{d}) > 0$ (otherwise consider $-p$), hence $p(\mathbf{e}) > 0$, too. Now, we use again an argument of continuity. Let $i := \sqrt{-1}$ be the imaginary number. We are going to show

$$\forall \alpha > 0: \ \forall s \geq 0: \ \forall t \in \mathbb{C}: \ p(\alpha i \mathbf{d} + t\mathbf{e} + s\mathbf{x}) = 0 \Rightarrow \operatorname{Im}(t) < 0. \tag{1.5}$$

Assume this statement is true (we are going to show this later on in this proof), i.e. all roots of $p(\alpha i \mathbf{d} + T\mathbf{e} + \mathbf{x})$ have negative imaginary part regardless the value of α. Now, we consider the limit value of the roots for α going to 0. The roots of the polynomial vary continuously with α, therefore all roots of $p(\mathbf{x} + T\mathbf{e}) = \lim_{\alpha \to 0} p(\alpha i \mathbf{d} + T\mathbf{e} + \mathbf{x})$ have non-positive imaginary part. The univariate polynomial $p(\mathbf{x} + T\mathbf{e})$ has only real coefficients, which means that all non-real roots of this polynomial appear in pairs of conjugates, i.e. if t is a root of the polynomial $p(\mathbf{x} + T\mathbf{e})$ with $\operatorname{Im}(t) \neq 0$, the complex conjugate $\bar{t}$ of t is a root of $p(\mathbf{x} + T\mathbf{e})$ as well. As we have seen, no roots of $p(\mathbf{x} + T\mathbf{e})$ have positive imaginary part, hence all roots must be real, which was the statement we wanted to show.

It remains to show the statement of (1.5). In order to do this, we fix some arbitrary $\alpha > 0$. In the case $s = 0$, we get for any $t \in \mathbb{C}$ with $p(\alpha i \mathbf{d} + t\mathbf{e}) = 0$:

$$0 = t^m p\left(\mathbf{e} + \frac{\alpha i}{t}\mathbf{d}\right).$$

Since p is hyperbolic in direction $\mathbf{d}$ by assumption, and $\mathbf{e} \in \mathbb{R}^n$, any root $\frac{\alpha i}{t}$ has to be real. Let us define $y := \frac{\alpha i}{t} \in \mathbb{R}$ to be such a root. By Proposition 1.2.12 (iii) it follows that $y < 0$. Hence $t = \frac{\alpha i}{y} \in i\mathbb{R}$ with $y < 0$ and $\alpha > 0$, which shows that $\operatorname{Im}(t) = \frac{\alpha}{y} < 0$. This is what we wanted to show.

Now assume, there is a $s > 0$ such that there is a zero t of the polynomial $p(\alpha i \mathbf{d} + T\mathbf{e} + s\mathbf{x})$ with $\operatorname{Im}(t) \geq 0$. Since this roots are continuous in s, there would be a $s' \in (0, s)$ such that $p(\alpha i \mathbf{d} + T\mathbf{e} + s'\mathbf{x})$ has a real root t'. This means

$$p(\alpha i \mathbf{d} + t'\mathbf{e} + s'\mathbf{x}) = 0,$$

which implies that αi is a root of the polynomial $p(T\mathbf{d} + (t'\mathbf{e} + s'\mathbf{x}))$. Since p is hyperbolic in direction $\mathbf{d}$ and $t'\mathbf{e} + s'\mathbf{x} \in \mathbb{R}^n$, the univariate polynomial $p(T\mathbf{d} + (t'\mathbf{e} + s'\mathbf{x}))$ has only real roots. This is a contradiction to αi being a root.

It remains to show the equality of the open hyperbolicity cones. This follows from the presentation (ii) of the hyperbolicity cone in Proposition 1.2.12. $\square$

1.2.15 Corollary. *[Ren06, Corollary 4]. For every $\mathbf{e} \in \Lambda(p, \mathbf{d})$, and for every point $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial $p(\mathbf{x} + T\mathbf{e})$ has only real roots.*

Now, we are able to show the main result of this section, which is that the hyperbolicity cones are convex. We already showed that the hyperbolicity cone is star-shaped with respect to the hyperbolic direction $\mathbf{d}$. So it is sufficient to show that it is star shaped in every direction $\mathbf{x}$ for a point $\mathbf{x}$ from the hyperbolicity cone.

1.2.16 Theorem. *[Ren06, Theorem 2]. All open hyperbolicity cones are convex.*

Proof. Let $p \in \mathbb{R}[\mathbf{X}]$ be a polynomial, hyperbolic in direction $\mathbf{d} \in \mathbb{R}^n$. For $\mathbf{x}, \mathbf{y} \in \Lambda(p, \mathbf{d})$ we only need to show that $\mathbf{x} + \mathbf{y} \in \Lambda(p, \mathbf{d})$ since we have already shown in Proposition 1.2.4 that the open hyperbolicity cone is closed under multiplication with positive scalars. Since $\mathbf{y}$ is in the open hyperbolicity cone, p is hyperbolic in direction $\mathbf{y}$ and $\Lambda(p, \mathbf{d}) = \Lambda(p, \mathbf{y})$ (see 1.2.14). WLOG we assume that $\mathbf{y} = \mathbf{d}$. Corollary 1.2.7 implies that the line between $\mathbf{x}$ and $\mathbf{d}$ is in the hyperbolicity cone included. Hence the cone in convex. □

1.2.17 Corollary. $\overline{\Lambda}(p, \mathbf{d})$ *is a convex cone.*

1.3 Derivatives of hyperbolic polynomials

We are not only interested in the properties of hyperbolic polynomials and their eigenvalues, but also how to get new hyperbolic polynomials out of the known ones, i.e. how to construct new hyperbolic polynomials. An obvious way is to multiply two hyperbolic polynomials. As a corollary of this section, we will see that all elementary symmetric polynomials are hyperbolic polynomials. We start with some easy examples and determine their hyperbolicity cones.

1.3.1 Lemma. *[KPV15, Lemma 2.2]. Let p, q be two homogeneous polynomials in $\mathbb{R}[\mathbf{X}]$ and $\mathbf{d} \in \mathbb{R}^n$ any direction. The product $p \cdot q$ is hyperbolic in direction $\mathbf{d}$ if and only if both polynomials p and q are hyperbolic in direction $\mathbf{d}$. In this case, $\Lambda(p \cdot q, \mathbf{d}) = \Lambda(p, \mathbf{d}) \cap \Lambda(q, \mathbf{d})$.*

Proof. Directly from the factorisation of $p(\mathbf{x} + T\mathbf{d})$ and $q(\mathbf{x} + T\mathbf{d})$ for all $\mathbf{x} \in \mathbb{R}^n$. □

Furthermore, there is another possibility to construct some hyperbolic polynomials. For example through derivation. For this reason, we need to introduce the formal directional derivation.

1.3.2 Definition. Let R be a commutative ring. For any polynomial $p = \sum_{\substack{\alpha \in \mathbb{N}_0^n, \\ |\alpha| \leq m}} c_\alpha \mathbf{X}^\alpha \in R[\mathbf{X}]$, we define the *(formal) partial derivative* $\frac{\partial}{\partial X_k} p$ with respect to the variable X_k for a $k \in \{1, \dots, n\}$ as

$$\frac{\partial}{\partial X_k} p := \sum_{\substack{\alpha - e_k \in \mathbb{N}_0^n, \\ |\alpha| \leq m}} \alpha_k c_\alpha \mathbf{X}^{\alpha - \mathbf{e}_k},$$

where $(\mathbf{e}_k)_{k \in [n]}$ denote the standard basis vectors of R^n.

With the partial derivative, we are now able to define the *(formal) directional derivative* $D_{\mathbf{v}} p$ of the polynomial p in direction $\mathbf{v} = (v_1, \dots, v_n) \in R^n$.

$$D_{\mathbf{v}} p = \sum_{k=1}^{n} v_k \frac{\partial}{\partial X_k} p$$

As usual, we define the k-th derivative $D_{\mathbf{v}}^{(k)}p$ recursive through $D_{\mathbf{v}}^{(0)}p := p$ and $D_{\mathbf{v}}^{(k+1)}p := D_{\mathbf{v}}(D_{\mathbf{v}}^{(k)}p)$ for all $k \in \mathbb{N}_0$.

1.3.3 Remark. For any univariate polynomial $p \in R[T]$ for any ring R, the derivative p' denotes the usual one dimensional (formal) derivative, which is the same as the directional derivative in direction $v = 1 \in R$.

With this definition of the formal derivative, it is possible to prove some well-known theorems from calculus as Rolle's Theorem. For more details and the proof, see [Pri13, p.30].

1.3.4 Theorem (Rolle's Theorem)**.** *Let F be any real closed field and $p \in F[T]$ any univariate polynomial over F. For two successive zeros $a, b \in F$ with $a \leq b$ of p there exists a point c in the interval (a, b) such that $p'(c) = 0$.*

Back to our construction of new hyperbolicity cones.

1.3.5 Proposition. *Let $p \in \mathbb{R}[\mathbf{X}]$ be hyperbolic in direction $\mathbf{d} \in \mathbb{R}^n$ of degree $m \in \mathbb{N}$. The directional derivative $D_{\mathbf{d}}p$ is hyperbolic in the same direction as p itself. For the hyperbolicity cones of p and $D_{\mathbf{d}}p$, we get the inclusion $\overline{\Lambda}(p, \mathbf{d}) \subseteq \overline{\Lambda}(D_{\mathbf{d}}p, \mathbf{d})$.*

Proof. The proof of this proposition is an easy consequence from Rolle's Theorem. Let $p \in \mathbb{R}[\mathbf{X}]$ be a polynomial, hyperbolic in direction $\mathbf{d} \in \mathbb{R}^n$ such that it is homogeneous and $p = \sum_{\substack{\alpha \in \mathbb{N}_0^n, \\ |\alpha| = m}} c_\alpha \mathbf{X}^\alpha$.
By the definition of hyperbolicity, this means that for every $\mathbf{x} \in \mathbb{R}^n$ all roots of $p(\mathbf{x} + T\mathbf{d})$ are real. Let $\mathbf{x}$ be an arbitrary point in $\mathbb{R}^n$. We need to show, that $(D_{\mathbf{d}}p)(\mathbf{x} + T\mathbf{d})$ has only real zeros. By the definition of the formal derivative, it follows

$$
\begin{aligned}
(D_{\mathbf{d}}p)(\mathbf{x} + T\mathbf{d}) &= \left(\sum_{k=1}^{n} d_k \frac{\partial}{\partial X_k} p \right) (\mathbf{x} + T\mathbf{d}) \\
&= \left(\sum_{k=1}^{n} d_k \sum_{\substack{\alpha - e_k \in \mathbb{N}_0^n, \\ |\alpha| = m}} \alpha_k c_\alpha \mathbf{X}^{\alpha - e_k} \right) (\mathbf{x} + T\mathbf{d}) \\
&= \sum_{k=1}^{n} d_k \sum_{\substack{\alpha - e_k \in \mathbb{N}_0^n, \\ |\alpha| = m}} \alpha_k c_\alpha (\mathbf{x} + T\mathbf{d})^{\alpha - e_k} \\
&= \sum_{k=1}^{n} d_k \frac{\partial}{\partial (x_k + T d_k)} p(\mathbf{x} + T\mathbf{d}) \\
&= (p(\mathbf{x} + T\mathbf{d}))'.
\end{aligned}
$$

The last equality holds because of the product- and chain-rule for the one-dimensional formal derivative. In the case $m = 1$ the derivative $(p(\mathbf{x} + T\mathbf{d}))'$ has degree $m = 0$, so it is hyperbolic in direction $\mathbf{d}$ (see Example 1.1.9) and the set-inclusion of the hyperbolicity cones is trivial.

For $m > 1$, we are able to apply Rolle's Theorem 1.3.4. This says that the roots of $(p(\mathbf{x}+T\mathbf{d}))'$ are those separating the ones of $p(\mathbf{x}+T\mathbf{d})$. So if $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m$ are the zeros of $p(\mathbf{x}+T\mathbf{d})$

(all real because p is hyperbolic in direction $\mathbf{d}$), Rolle's Theorem says there are $m-1$ zeros $\beta_1, \ldots, \beta_{m-1}$ of $(p(\mathbf{x}+T\mathbf{d}))'$ such that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_{m-1}$ and $\alpha_j \leq \beta_j \leq \alpha_{j+1}$ for all $j \in [m-1]$.

Figure 1.4: Roots of $p(\mathbf{x}+T\mathbf{d})$ and $(p(\mathbf{x}+T\mathbf{d}))'$.

So all $m-1$ zeros of $(p(\mathbf{x}+T\mathbf{d}))'$ are real. This argument also shows that $\Lambda(p, \mathbf{d}) \subseteq \Lambda(D_{\mathbf{d}}p, \mathbf{d})$. If we take a point $\mathbf{x} \in \Lambda(p, \mathbf{d})$, the eigenvalues $\lambda_k(\mathbf{d}, \mathbf{x})$ are positive. Hence the roots of $p(\mathbf{x}+T\mathbf{d})$ are negative and so are the roots of $(p(\mathbf{x}+T\mathbf{d}))'$ as seen before. So the eigenvalues of $\mathbf{x}$ in direction $\mathbf{d}$ with respect to $D_{\mathbf{d}}p$ are positive, which is the condition for $\mathbf{x}$ to be a point in $\Lambda(D_{\mathbf{d}}p, \mathbf{d})$. □

1.3.6 Proposition. *[Går59, p.3]. For any in direction $\mathbf{d} \in \mathbb{R}^n$ hyperbolic polynomial $p \in \mathbb{R}[\mathbf{X}]$ of degree m, the polynomials $p_k \in \mathbb{R}[\mathbf{X}]$ $(k = 1, \ldots, m)$ defined by $p(\mathbf{X}+T\mathbf{d}) = \sum_{k=0}^{m} T^{(m-k)} p_k \in \mathbb{R}[X_1, \ldots, X_n][T]$ are hyperbolic in direction $\mathbf{d}$.*

Proof. First, we want to mention that the polynomials p_k are well-defined since we consider $p(\mathbf{X}+T\mathbf{d})$ as a univariate polynomial in $\mathbb{R}[\mathbf{X}][T]$ such that the coefficients $p_k \in \mathbb{R}[\mathbf{X}]$ of the univariate polynomial in T are unique.

As we have seen in the proof of Proposition 1.3.5, it holds $(p(\mathbf{x}+T\mathbf{d}))^{(k)} = D_{\mathbf{d}}^{(k)} p(\mathbf{x}+T\mathbf{d})$ for any $\mathbf{x} \in \mathbb{R}^n$ and $k \in \mathbb{N}_0$. (We have seen this equation only for the case $k=1$. The case $k=0$ is trivial and the more general case for an arbitrary $k \in \mathbb{N}$ follows directly by induction). By repeated application of Proposition 1.3.5, all derivatives $D_{\mathbf{d}}^{(k)}p$, $k \in \mathbb{N}_0$, of p are hyperbolic in direction $\mathbf{d}$. Hence for any $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial $(p(\mathbf{x}+T\mathbf{d}))^{(k)}$ has only real roots. Moreover, the k-th derivative of $p(\mathbf{X}+T\mathbf{d})$ as a univariate polynomial in the variable T and evaluated at the point 0 is

$$(p(\mathbf{X}+T\mathbf{d}))^{(k)}\Big|_{T=0} = k! p_{m-k}$$

Now assume, for one $k \in [m] \cup \{0\}$ the polynomial p_{m-k} is not hyperbolic in direction $\mathbf{d}$, hence there is a point $\mathbf{x} \in \mathbb{R}^n$ and a $t_0 \in \mathbb{C}$ with $\mathrm{Im}(t_0) \neq 0$ such that $p_k(\mathbf{x}+t_0\mathbf{d}) = 0$. This implies

$$\begin{aligned} 0 = k! p_{m-k}(\mathbf{x}+t_0\mathbf{d}) &= (p(\mathbf{X}+T\mathbf{d}))^{(k)}\Big|_{T=0} (\mathbf{x}+t_0\mathbf{d}) \\ &= \Big((D_{\mathbf{d}}^{(k)}p)(\mathbf{X}+T\mathbf{d})\Big)\Big|_{T=0} (\mathbf{x}+t_0\mathbf{d}) \\ &= (D_{\mathbf{d}}^{(k)}p)(\mathbf{x}+t_0\mathbf{d}). \end{aligned}$$

Hence $D_{\mathbf{d}}^{(k)}(\mathbf{x}+T\mathbf{d})$ has a root t_0 with $\mathrm{Im}(t_0) \neq 0$, which is a contradiction to the fact that $D_{\mathbf{d}}^{(k)}p$ is hyperbolic in direction $\mathbf{d}$. □

1.3.7 Definition. [Brä13, p.2 and p.4]. For $S \subseteq [n]$, we define the k-th elementary symmetric polynomial for $k \in \mathbb{N}_0$ in the $|S|$ variables $(X_j)_{j \in S}$ as

$$\sigma_k(S) := \sum_{\substack{T \subseteq S \\ |T|=k}} \prod_{j \in T} X_j \in \mathbb{R}[\mathbf{X}].$$

We write $\sigma_k := \sigma_k([n])$ for all $k \in [n] \cup \{0\}$.

1.3.8 Remark. The k-th elementary symmetric polynomial σ_k is always a homogeneous polynomial of degree k. We also defined elementary symmetric polynomial σ_0. It is $\sigma_0 = 1$.Furthermore, $\sigma_k(S) = 0$ for any $k > |S|$ and any $S \subseteq [n]$.

1.3.9 Proposition. *All elementary symmetric polynomials are hyperbolic in direction* $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$.

Proof. As we have seen in Example 1.1.9 (i), the polynomial $p = \prod_{k=1}^{n} X_k$ is hyperbolic in any direction $\mathbf{d} \in \mathbb{R}^n$ with $p(\mathbf{d}) \neq 0$. For this proposition we consider $\mathbf{d} = (1, \ldots, 1) \in \mathbb{R}^n$. Now consider the polynomial $p(\mathbf{X} + T\mathbf{d}) \in \mathbb{R}[X_1, \ldots, X_n][T]$ as a univariate polynomial in the ring $R[T]$ with the ring $R := \mathbb{R}[\mathbf{X}]$. The coefficients of this polynomial as a univariate polynomial (elements of $R = \mathbb{R}[\mathbf{X}]$) are exactly the elementary symmetric polynomials $\sigma_k \in \mathbb{R}[\mathbf{X}]$ (for all $k = 0, 1, \ldots, n$). Hence all elementary symmetric polynomials are hyperbolic by Proposition 1.3.6. $\square$

Chapter 2

Graphs and Digraphs

In this chapter, we shortly introduce graphs, the undirected version, and some important statements about graphs, trees and especially spanning trees of graphs. Afterwards, we define the directed version of graphs, called digraphs and considered as graphs with darts instead of just (undirected) edges. The directed analogue of trees is called arborescences. Those arborescences consist of a vertex called root such that all darts are diverging from this root.

2.1 Graphs

A graph consists of a finite set of vertices, mostly drawn as points, and a finite set of edges, drawn as lines between the vertices. The definition which suits for our interests, is often called multi-graph because it is possible to have several edges between any two vertices. Furthermore, we do not allow loops, edges between a vertex and itself. The formal definition is:

2.1.1 Definition. A *graph* $G = (V_G, E_G, \epsilon_G)$ consists of two finite sets V_G and E_G, where V_G is the *vertex-set* and E_G the *edge-set* of the graph G. Furthermore, there is a function

$$\epsilon_G : E_G \to \{\{x, y\} : x, y \in V_G \wedge x \neq y\},$$

which assigns to every edge $e \in E_G$ an unordered pair of vertices, the two to e incident vertices. If there are edges $e_1, e_2 \in E_G$ such that $\epsilon_G(e_1) = \epsilon_G(e_2)$, we say the graph contains a *multi-edge* between the two incident vertices.

If not otherwise specified, E_G and V_G will always denote the set of edges and vertices of a graph G. In this whole chapter, $n \in \mathbb{N}_0$ denotes always the number of vertices in a graph.

2.1.2 Example. Let us consider the graph $G = (V_G, E_G, \epsilon_G)$ on the vertices $V_G = \{v_1, \ldots, v_7\}$ and with edges $E = \{e_1, \ldots, e_7\}$. If we draw a graph, we consider the vertices to be nodes, and the edges lines or arcs between the two incident vertices given by the function ϵ_G. In this example, the function ϵ is defined by

$\epsilon_G(e_1) = \epsilon_G(e_2) = \{v_1, v_2\}$,
$\epsilon_G(e_3) = \{v_2, v_3\}$,
$\epsilon_G(e_4) = \{v_1, v_4\}$,
$\epsilon_G(e_5) = \{v_3, v_5\}$,
$\epsilon_G(e_6) = \{v_4, v_5\}$,
$\epsilon_G(e_7) = \{v_4, v_6\}$.

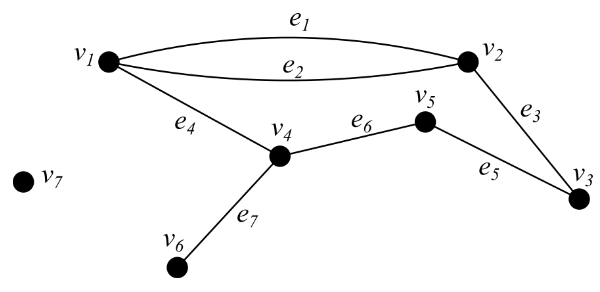


Figure 2.1: One possibility to draw the Graph G.

There are a lot of possibilities to draw a graph. One possibility to draw the graph G as defined above is shown in Figure 2.1.

2.1.3 Remark. A graph G has no multi-edges if and only if ϵ_G is injective. If ϵ_G is injective, we say G is a *simple graph.*

We continue with some elementary definitions belonging to a graph.

2.1.4 Definition. Let $G = (V, E, \epsilon)$ be a graph.

(a) If there is an edge $e \in E$ between two vertices $v, w \in V$, which means $\epsilon(e) = \{v, w\}$, the two vertices v and w are said to be *neighbours.*

(b) The *degree* of a vertex $v \in V$ is the number of incident edges, denoted by $\deg(v)$ and defined through $\deg(v) := |\{w \in V : w \text{ is a neighbour of } v\}|$.

(c) If $\deg(v) = 0$ for any vertex $v \in V$, we say v is *isolated* and if $\deg(v) = 1$, the vertex v is called *leaf.*

2.1.5 Definition. Let $G = (V_G, E_G, \epsilon_G)$, $H = (V_H, E_H, \epsilon_H)$ be graphs. H is said to be a *subgraph* of G, denoted by $H \subseteq G$, if $V_H \subseteq V_G$, $E_H \subseteq E_G$ and $\epsilon_H = \epsilon_G|_{E_H}$.

Furthermore H is called a *spanning subgraph* of G if H is a subgraph of G such that $V_H = V_G$.

2.1.6 Remark. For every Graph G the empty graph $(\emptyset, \emptyset, \epsilon)$ and G itself are subgraphs of G .

If we follow the edges of a drawn graph, it is possible to go from one vertex to another vertex just using the edges appearing in the considered graph. More precisely, we define a path in a graph as follows.

2.1.7 Definition. Let $G = (V, E, \epsilon)$ be a graph. A *path* P of length $k \in \mathbb{N}_0$ in the graph G is a sequence $v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$ of pairwise distinct vertices $v_0, v_1, \ldots, v_k \in V$ and pairwise distinct edges $e_1, e_2, \ldots, e_k \in E$ such that $\{v_{i-1}, v_i\} = \epsilon(e_i)$ for all $i \in [k]$.

A *cycle* C in a graph is similar to a path a sequence $v_k, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$ of pairwise distinct edges and pairwise distinct vertices of G such that $\{v_{i-1}, v_i\} = \epsilon(e_i)$ for all $i \in [k]$ and $v_0 := v_k$.

2.1.8 Remark. Cycles and paths in a graph G could be considered as a subgraph of G itself. Say $v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$ is a path or a cycle. Then the graph (V, E, ϵ) consisting of the vertices $V = \{v_0, v_1, \ldots, v_k\}$ and the edges $E = \{e_1, e_2, \ldots, e_k\}$ and with the function ϵ of incident vertices defined as $\epsilon := (\epsilon_G)|_E$ is a subgraph of G.

2.1.9 Example. In the graph shown in Figure 2.1, the neighbours of the vertex v_4 are v_1, v_5, v_6. Hence the degree of v_4 is $\deg(v_4) = |\{v_1, v_5, v_6\}| = 3$. The vertex v_6 is a leaf because it has only one neighbour, namely v_4. An example for an isolated vertex is v_7 since there are no incident edges to this vertex.

The sequence $v_6, e_7, v_4, e_6, v_5, e_5, v_3$ is a path P in G in between the two vertices v_6 to v_3. A cycle C is for example v_1, e_1, v_2, e_2, v_1.

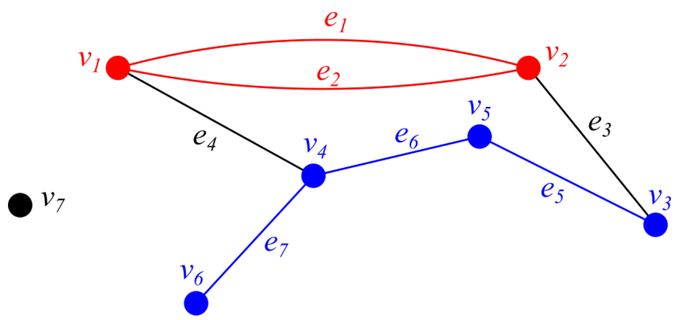


Figure 2.2: A path P in the graph G (marked in blue) defined in Figure 2.1 and a cycle C (marked in red).

2.1.10 Example (Special graphs)**.** In this example we will mention some special graphs without multi-edges, i.e. ϵ is injective.

(i) The Empty Graph $E_n = (V, E, \epsilon)$ on $|V| = n$ vertices, is a graph without edges. Hence $E = \emptyset$.

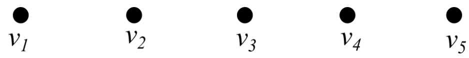


Figure 2.3: Empty graph E_5 on the five vertices $V = \{v_1, v_2, v_3, v_4, v_5\}$.

(ii) $P_n = (V, E, \epsilon)$ is the path on $|V| = n$ vertices just consisting of a path from the first to the last vertex. There are $n-1$ edges, with $\epsilon(E) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\}$ assuming $V = \{v_1, v_2, \ldots, v_n\}$.

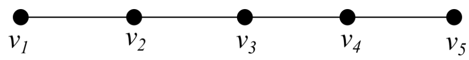


Figure 2.4: Path P_5 on the five vertices $V = \{v_1, v_2, v_3, v_4, v_5\}$.

(iii) The graph C_n is a cycle on the n vertices $v_1, \ldots, v_n$ with

$$\epsilon_{C_n}(E_{C_n}) = \{\{v_i, v_{i+1}\} : i \in [n-1]\} \cup \{\{v_n, v_1\}\}.$$

(iv) The Complete Graph K_n on n vertices consists of all possible edges between the vertices (every pair of vertices of the graph is exactly once connected by a graph, no multiple edges).

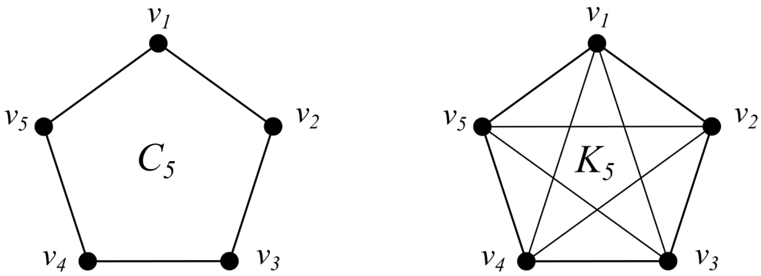


Figure 2.5: Cycle C_5 (left) and Complete Graph K_5 (right), both on the five vertices $V = \{v_1, v_2, v_3, v_4, v_5\}$.

2.1.11 Definition. Two vertices $v, w \in V_G$ in a Graph G are said to be *connected* if there is a path from v to w. A graph G itself is called *connected* if each pair of vertices of the graph is connected. Otherwise, we call G *disconnected*. A *connected component* of a graph G is a connected subgraph such that no other vertex of G is connected to one of the vertices of the connected component.

Later on, we need the following notations.

2.1.12 Definition. For any Graph $G = (V, E, \epsilon)$ and an edge $e \in E$ of G, we write

$$G - e := (V, E \backslash \{e\}, \epsilon|_{E \backslash \{e\}})$$

for the graph G without the edge e.

2.2 Trees

In this section, we consider a special type of graphs, called trees. There are multiple ways to define a tree. We will use the following definition, but soon we will see some equivalent conditions.

2.2.1 Definition. A *tree* is a connected graph without cycles.

2.2.2 Example.

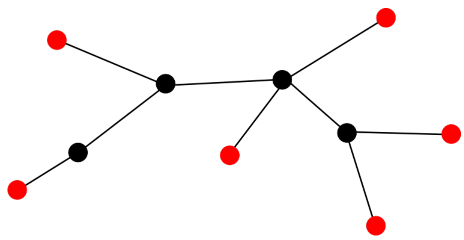


Figure 2.6: Tree on ten vertices, with nine edges and six leafs (marked in red).

2.2.3 Remark. (a) The empty graph $(\emptyset, \emptyset, \epsilon)$ is a tree.

(b) A tree never contains multi-edges. Otherwise there would exist a cycle. So the function ϵ_T is injective for any tree T. Hence we are in the case of a simple graph, defined in 2.1.3.

In the following proposition, we put down some characterisations of a tree, equivalent to the definition above. There are even more equivalent statements, but these are sufficient for this thesis.

2.2.4 Proposition. *Let $T = (V, E, \epsilon)$ be a graph containing at least one vertex. The following conditions are equivalent:*

(i) T is a tree.

(ii) T is connected and $|V| = |E| + 1$.

(iii) G contains no cycle and $|V| = |E| + 1$.

(iv) Any two vertices in T are connected by a unique path in T.

(v) G is connected, but after removing an edge the graph is disconnected.

Proof. See [GY98, Theorem 3.1.11]. □

2.2.5 Definition. If H is a spanning subgraph of a graph G and H is a tree, we say it is a *spanning tree* of G.

2.2.6 Proposition. *A graph has a spanning tree if and only if the graph is connected.*

Proof. See [Tut84, Theorem 1.36]. □

2.2.7 Proposition. *Let G be a simple graph on $n \geq 2$ vertices and with $m \in \mathbb{N}_0$ edges.*

(a) If $m < n - 1$, G is disconnected.

(b) If $m > \binom{n-1}{2}$, G is connected ([GY98, Corollary 3.1.10]).

Proof. (a) If G was connected, G would contain a spanning tree T with $n - 1$ edges (2.2.6 and 2.2.4). Since the edges of the spanning tree are a subset of the edges of G, G would contain at least $n - 1$ edges.

(b) We show the contraposition: If G is disconnected, the number of edges in G is smaller or equal than $\binom{n-1}{2}$.

If G is disconnected, the graph has at least two connected components. The most edges appear if there are only two connected components and each of them is a Complete Graph. Say K_{n_1} and K_{n_2} for $n_1, n_2 \in \mathbb{N}$ are the two connected components of G with $n = n_1 + n_2$. The number of edges in the Complete Graph K_{n_i} is $\binom{n_i}{2}$ for $i = 1, 2$. So number of edges appearing in the graph G is

$$\binom{n_1}{2} + \binom{n_2}{2}.$$

We want to show that the number of edges appearing is maximal if $n_1 = 1$ (or analogue $n_2 = 1$). So we need to show that

$$\binom{n_1}{2} + \binom{n_2}{2} \leq \binom{n-1}{2} + \binom{1}{2} = \binom{n-1}{2}.$$

This follows from

$$\begin{aligned} & n_1 n_2 - n_1 - n_2 + 1 = (n_1 - 1)(n_2 - 1) \geq 0 \\ \Leftrightarrow \quad & 2n_1 n_2 - 2n_1 - 2n_2 + 2 \geq 0 \\ \Leftrightarrow \quad & n_1^2 + 2n_1 n_2 + n_2^2 - 3n_1 - 3n_2 + 2 \geq n_1^2 - n_1 + n_2^2 - n_2 \\ \Leftrightarrow \quad & (n-1)(n-2) \geq n_1(n_1 - 1) + n_2(n_2 - 1) \end{aligned}$$

□

A well known theorem is the formula of Cayley to count the number of spanning trees in the Complete Graph on n vertices. There are multiple ways to prove this formula. A very common one uses the Prüfer-sequence, which we do not introduce in this work. For the details of the proof see [GY98, Theorem 4.4.4]. Later in this thesis, we will see another possibility to prove this formula, using the Matrix-Tree Theorem.

2.2.8 Theorem (Cayley's Formula)**.** *The Complete Graph K_n on n vertices has n^{n-2} spanning trees.*

2.2.9 Example. Using Cayley's Formula, we are able to calculate the number of spanning trees of the Complete Graph K_4, which is $4^{4-2} = 16$. In the same way, we know that the number of spanning trees of K_3 is $3^{3-2} = 3$.
But it is not only possible to calculate the number of spanning trees of a Complete Graph but we are also able to calculate the number of spanning trees of a graph consisting only of Complete Graphs connected by a single edge such as the one in the following figure.

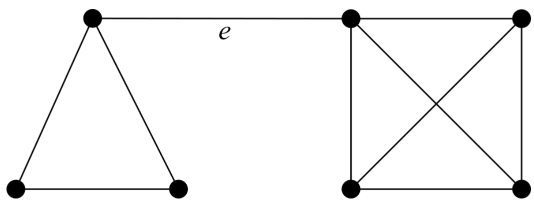


Figure 2.7: Graph G consisting of two Complete Graphs K_3 and K_4 as a subgraph connected by a single edge e.

The number of spanning trees of the graph drawn in Figure 2.7 is $3 \cdot 16 = 48$ because any spanning tree of this graph includes the edge e connecting both complete subgraphs K_3 and K_4.

2.3 Digraphs

Cayley's Formula is very interesting, but we also want to count spanning trees of an arbitrary graph, not only of the Complete Graph. This is the motivation behind the Matrix-Tree Theorem, which will be proved in the next chapter. To formulate and prove this theorem, we need to prove this theorem first in the version for digraphs. In order to do this, we need the basic definitions considering digraphs.

The term 'digraph' comes from the term 'directed graph' and is a shortcut for this. Still it is very common in the literature to use this shortcut.

2.3.1 Definition. A *digraph* is a triple $\Gamma = (V_\Gamma, D_\Gamma, o_\Gamma)$ consisting of a finite vertex-set V, a finite dart-set D and an orientation function $o : D \to V \times V$ which assigns to every dart $d \in D$ of the digraph Γ a tuple of two vertices, the first where the dart starts (the *source-vertex*) and the second one determines the endpoint (the *target-vertex*). Hence, we can consider the functions $s : D \to V$ and $t : D \to V$ assigning the source- and target-vertex to every dart. We do not allow loops, so for every dart $d \in D_\Gamma$ the start and the end vertex must be distinct, i.e. $\forall d \in D_\Gamma : s(d) \neq t(d)$.

We say a digraph Γ contains a multi-dart if there are two different darts d_1, d_2 in Γ with the same start and the same endvertex. So $o(d_1) = o(d_2)$.

2.3.2 Remark. If not mentioned otherwise, V_Γ, D_Γ and o_Γ denote the vertex-, dart-set and the orientation function of a digraph Γ. The orientation function o_Γ is injective if and only if the digraph Γ does not contain any multi-darts.

A digraph is something similar to a graph. There is one important difference. The 'edges' of a digraph have a direction and hence they are called darts.

2.3.3 Example. If we draw a digraph, we draw the element of the dart-set as darts. Each of the darts is going from its start-vertex to its end-vertex. In the following figure, the digraph Γ_1 is a digraph on the five vertices $V_1 = \{v_1, v_2, v_3, v_4, v_5\}$, and with six directed edges.

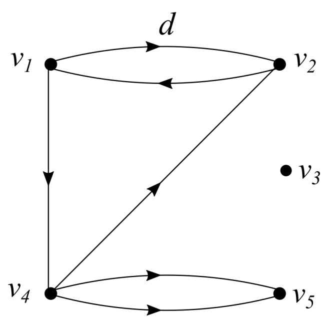


Figure 2.8: Digraph Γ_1 on the five vertices $V_1 = \{v_1, v_2, v_3, v_4, v_5\}$.

The orientation of the dart d is $o(d) = (v_1, v_2)$ since the dart goes from v_1 to v_2. Similar for the other darts drawn in the figure.

2.3.4 Example. Another example for a digraph is the *Complete Digraph on n vertices* consisting of all possible darts between the vertices without any multi-darts.

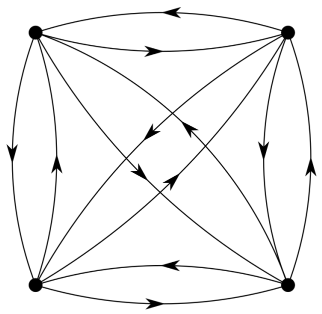


Figure 2.9: Complete digraph on four vertices.

Similar to the degree of an (undirected) graph (Definition 2.1.4 (b)), we now want to define the number of incoming darts and the number of outgoing darts:

2.3.5 Definition. Let $\Gamma = (V, D, o)$ be a digraph. To every vertex $v \in V$, we define the *incoming degree* as the number of darts in Γ with target-vertex v:

$$\operatorname{indeg}(v) := \operatorname{indeg}_\Gamma(v) := |\{d \in D : \ t(d) = v\}|.$$

Analogously, we define the *outgoing degree* as

$$\operatorname{outdeg}(v) := \operatorname{outdeg}_\Gamma(v) := |\{d \in D : \ s(d) = v\}|.$$

2.3.6 Remark. In a digraph $\Gamma = (V, D, o)$, every dart has a start and an end-vertex such that the sum over all incoming degrees and the sum over all outgoing degrees coincides with the number of darts:

$$|D| = \sum_{v \in V} \operatorname{indeg}(v) = \sum_{v \in V} \operatorname{outdeg}(v).$$

2.3.7 Example. Consider again the digraph Γ_1 drawn in Figure 2.8. The vertex v_3 has incoming and outgoing degree zero. For v_4 the degrees are $\operatorname{indeg}(v_4) = 1$ and $\operatorname{outdeg}(v_4) = 3$.

2.3.8 Definition. A *subdigraph* $\Delta = (V_\Delta, D_\Delta, o_\Delta)$ of a digraph $\Gamma = (V_\Gamma, D_\Gamma, o_\Gamma)$ is a digraph itself such that

$$V_\Delta \subseteq V_\Gamma, \qquad D_\Delta \subseteq D_\Gamma \quad \text{and} \quad o_\Delta = o_\Gamma|_{D_\Delta}.$$

The last part guarantees that the orientation of the directed edges in the subdigraph Δ is the same as in Γ.

2.3.9 Example.

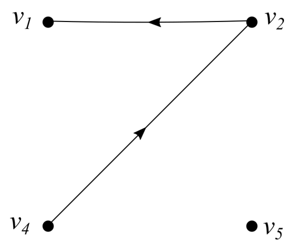


Figure 2.10: Subdigraph Δ_1 of the digraph Γ_1 in Figure 2.8.

For a graph, we defined a path to be a sequence of vertices and edges. We want to define something similar for digraphs. As already mentioned, the main difference between graphs and digraphs is the orientation of the edges. A path in a digraph can only use the edges in one direction. So a path from vertex v to vertex w has a direction and is not reversible.

2.3.10 Definition. A *(directed) path* P in a digraph $\Gamma = (V, D, o)$ of length $k \in \mathbb{N}_0$ is a sequence $v_0, d_1, v_1, d_2, v_2, \ldots, v_{k-1}, d_k, v_k$ of pairwise distinct vertices $v_0, v_1, \ldots, v_k \in V$ and darts $d_1, \ldots, d_k \in D$ such that $o(d_i) = (v_{i-1}, v_i)$ for all $i \in [k]$.

A *(directed) cycle* C in a digraph $\Gamma = (V, D, o)$ of length $k \in \mathbb{N}_0$ is a sequence of the form $v_k, d_1, v_1, d_2, v_2, \ldots, v_{k-1}, d_k, v_k$ with pairwise distinct vertices $v_1, \ldots, v_k \in V$ and darts $d_1, \ldots, d_k \in D$ such that $o(d_i) = (v_{i-1}, v_i)$ for all $i \in [k]$ and $v_0 := v_k$.

A *tour* T is a sequence $v_0, d_1, v_1, d_2, v_2, \ldots, v_{k-1}, d_k, v_0$ of vertices $v_0, \ldots, v_{k-1}$ and pairwise distinct darts $d_1, \ldots, d_k$ such that the start- and end-vertex are the same and $o(d_i) = (v_{i-1}, v_i)$ for all $i \in [k]$ and $v_k := v_0$.

2.3.11 Example.

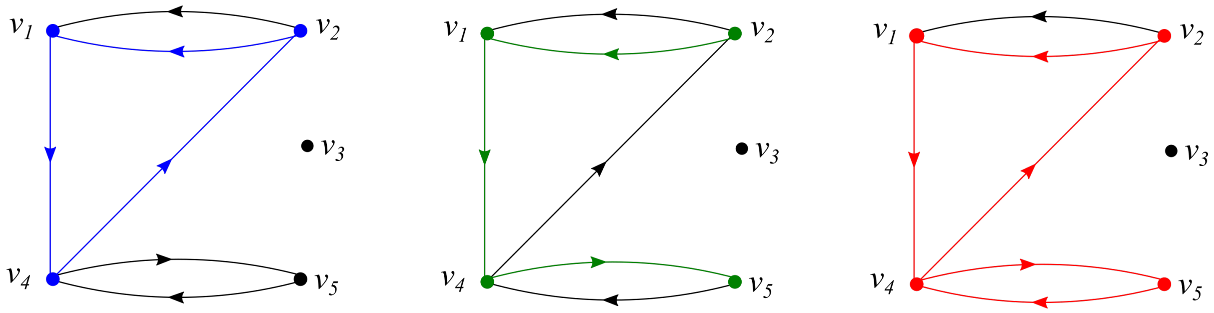


Figure 2.11: On the left-hand side, we see a cycle C in a digraph Γ (blue) and in the middle a path from v_2 to v_5 in Γ (green). On the right-hand side, there is a tour in red.

There are not only parallels between graphs and digraphs but it is also possible to construct a digraphs out of a given graph and vice versa.

2.3.12 Construction. *(a) Let $\Gamma = (V, D, o)$ be a digraph. The underlying (undirected) graph $U(\Gamma) = (V, D, \epsilon)$ is a graph with the same vertex-set V and the same edge-set D, but we remove the 'darts' on the edges and get undirected edges. So if φ is the function*

$$\varphi:\ V \times V \to \{\{x, y\} :\ x, y \in V\}, \quad (x, y) \mapsto \{x, y\} \tag{2.1}$$

assigning to an ordered pair of vertices the set of both entries of the ordered pair, we require ϵ to be $\epsilon = \varphi \circ o$.

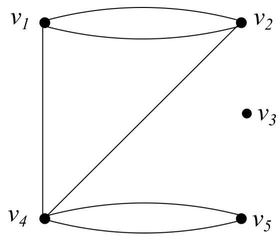


Figure 2.12: Underlying graph $U(\Gamma_1)$, for Γ_1 see Figure 2.8.

(b) *Conversely, there are multiple ways to construct a digraph out of a given graph $G = (V, E, \epsilon)$. The first one coming into mind is to assign to every undirected edge $e \in E$ an arbitrary orientation. So for the resulting digraph $\Gamma = (V, E, o)$, the orientation must fulfil $\varphi \circ o = \epsilon$. This construction is not unique, so we may get different digraphs with this version. Still the underlying graph $U(\Gamma)$ is always G itself for every possible digraph Γ.*

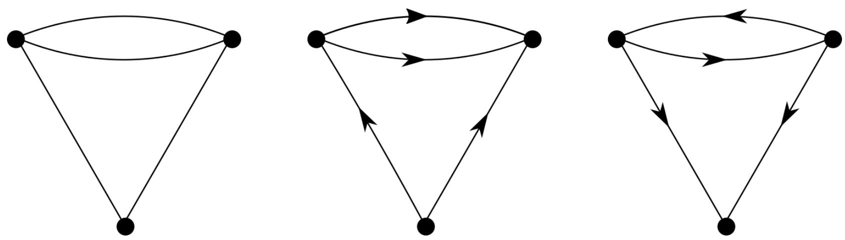


Figure 2.13: The leftmost graph is the undirected graph G. The other two graphs are two different possibilities for the orientation of the edges of G to get a digraph. Still these are not the only possibilities.

(c) *Another way coming into mind to construct a digraph $\Gamma = (V, D, o)$ from a given graph $G = (V, E, \epsilon)$ is to double the number of darts compared to the number of edges in G by setting $D := \{e^+ : e \in E\} \cup \{e^- : e \in E\}$. The vertex-set stays the same as in the given undirected graph. We define the orientation-function to be*

$$\begin{aligned} o:\ & D \to V \times V, \\ & e^+ \mapsto (u, v), \\ & e^- \mapsto (v, u) \qquad\qquad \textit{if } \epsilon(e) = \{v, u\}. \end{aligned}$$

It does not matter which dart has which direction, so we do not care about this. This digraph is called the equivalent digraph and is denoted by $\widetilde{G} = \Gamma$.

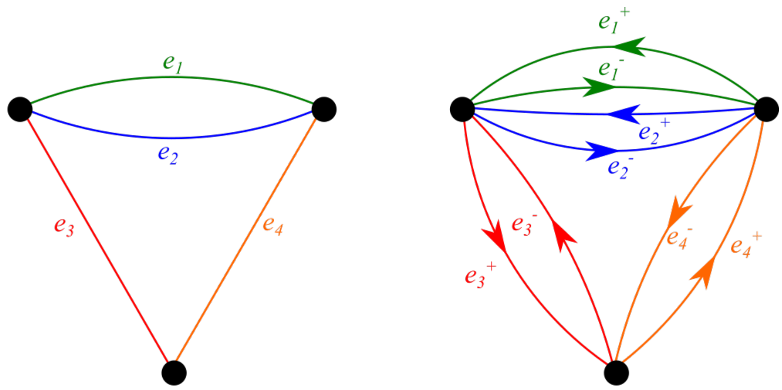


Figure 2.14: A graph G (left) and its equivalent digraph $\widetilde{G}$ (right).

This construction gives a unique digraph (up to the definition of e^+ and e^-). Only if there are no edges in the graph G, the underlying graph of $\widetilde{G}$ is the graph G itself.

For any graph, the digraph in (b) is always a subdigraph of the equivalent digraph.

2.4 Arborescences

In this section, we consider a directed analogue of trees. We want to define a digraph satisfying the conditions of a tree (or at least some of them), see 2.2.4. As far as possible, it should be a digraph without cycles and with one edge less than vertices such that the underlying graph is a tree.

This condition is satisfied if we use the construction of 2.3.12 (b). The problem of this version is, that in general there is no path with length longer than one.

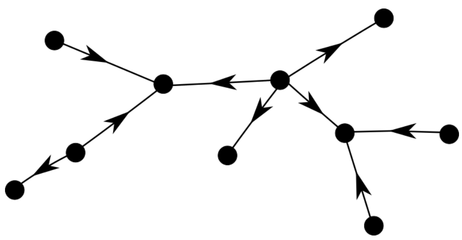


Figure 2.15: Tree from Figure 2.6 with orientation such that there is no path longer than length one.

If we want to have a path between any two vertices (see Proposition 2.2.4 (iv)), we need the equivalent digraph. On the other hand, if we use the equivalent digraph of a tree, there is always a directed cycle.

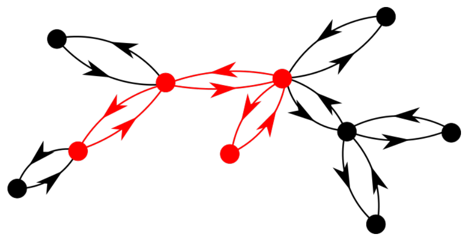


Figure 2.16: Equivalent Digraph of the tree shown in Figure 2.6 containing a cycle (red).

So let us summarize the aims of the following definition. We would like to define a digraph such that the underlying graph is a tree, but there should also be a possibility to go from one vertex to another. This is in general not possible, so we decrease the requirements. Instead of requesting a path between any two vertices, we only ask for a path from a fixed vertex r to any other vertex in the graph. This path is then unique as well.

2.4.1 Construction. *Consider an (undirected) tree $T = (V_T, E_T, \epsilon_T)$. The function ϵ_T is injective so we can neglect it. Let $r \in V_T$ be an arbitrary fixed vertex. For any edge $e \in E_T$ the graph $T - e$ (see Definition 2.1.12) consists of exactly two connected components (2.1.11), called R_e, S_e where R_e is the component with $r \in R_e$.*

Now, we want to construct a digraph (V_T, E_T, o_T) with orientation such that all darts are orientated away from r . For this, we assign to every edge e of the tree T with $\epsilon_T(e) = \{u, v\}$ the orientation $o_T(e) = (u, v)$ if $u \in R_e$ and $v \in S_e$.

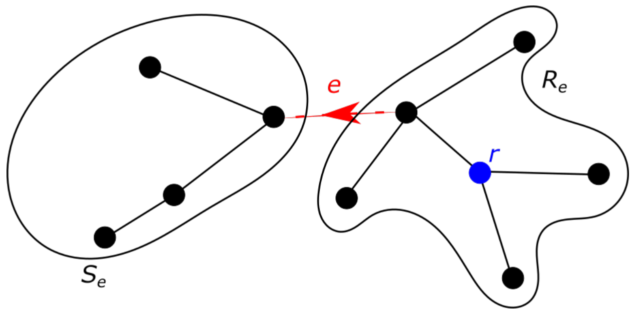


Figure 2.17: Tree with the two connected components R_e and S_e for the edge e and the resulting orientation of the dart e.

Preceding like this for every edge $e \in E$, we receive a digraph.

2.4.2 Definition. A digraph $A = (V_A, D_A, o_A)$ is called an *arborescence diverging from* $r \in V_A$ if there is a tree $T = (V_A, D_A, \epsilon_T)$ such that $o_A(d) = (u, v)$ for every edge $d \in D_A$ with $\epsilon_T(d) = \{u, v\}$ and $u \in R_d, v \in S_d$ (R_d and S_d as defined above in the construction 2.4.1).

The vertex $r \in V_A$ is called the *root* of the arborescence.

2.4.3 Example. A possible arborescence diverging from r is:

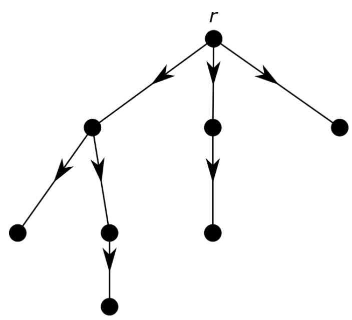


Figure 2.18: Arborescence diverging from a root r.

The root r plays an important role. Depending on the choice of r, the arborescence varies.

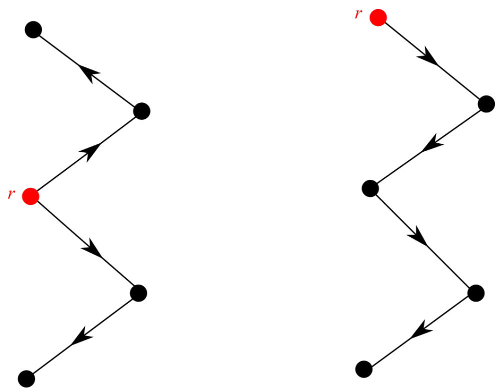


Figure 2.19: Two possible arborescences on the same underlying tree with different roots

2.4.4 Remark. Let us check if the requirements mentioned above are fulfilled by the definition of an arborescence diverging from r. The underlying graph is obviously a tree because we just assigned to every edge in a tree a direction. So there is no cycle and one dart less than vertices. Furthermore, there are no multi-darts in an arborescence. Otherwise the underlying graph would contain a cycle.

The construction of an arborescence guarantees that there is always a path from r to any other vertex because all darts are oriented away from r.

2.4.5 Remark. It is also possible to define an arborescence converging to a fixed vertex r instead of a diverging arborescence. But this is just the same digraph with all orientations reversed.

2.4.6 Lemma. *Let Γ be a digraph and $r \in V_\Gamma$ a vertex. The digraph Γ is an arborescence diverging from r if and only if Γ does not contain any cycle and indeg$(r) = 0$ as well as indeg$(v) = 1$ for every $v \in V_\Gamma \backslash \{r\}$.*

Proof. '$\Leftarrow$:' Let $\Gamma = (V, D, o)$ be a digraph without cycles and with some vertex $r \in V$ satisfying $\text{indeg}(r) = 0$ and $\text{indeg}(v) = 1$ for all other vertices $v \in V \backslash \{r\}$. The fact that Γ is without cycles implies that the underlying graph $U(\Gamma)$ is without cycles as well.

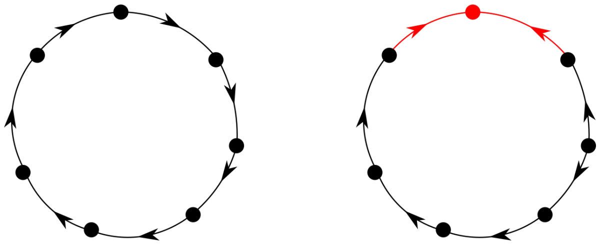


Figure 2.20: Possible digraphs with a cycle as underlying graph.

A cycle in $U(\Gamma)$ would imply that there is either a directed cycle in Γ or there is a vertex $v \in V$ with $\operatorname{indeg}(v) \geq 2$. In both cases we get a contradiction to the assumption.

Furthermore, the number of darts is

$$|D| \overset{2.3.6}{=} \sum_{v \in V} \operatorname{indeg}(v) = \underbrace{\operatorname{indeg}(r)}_{=0} + \sum_{v \in V \setminus \{r\}} \underbrace{\operatorname{indeg}(v)}_{=1} = |V| - 1.$$

By Proposition 2.2.4 $U(\Gamma)$ is a tree. It remains to show that the edges in Γ are orientated in the correct way.

Consider the vertex r. Let $N(v)$ denote the set of all neighbours of v for all $v \in V$ in $U(\Gamma)$. Since $\operatorname{indeg}(r) = 0$ all darts with r as a incident vertex must be orientated away from r to any of its neighbours. So for all $v \in N(r)$ the incoming degree is $\operatorname{indeg}(v) \geq 1$. By assumption it is exactly the degree is exactly one. This means that there is no other dart with v as target-vertex for all $v \in N(r)$. For any $v \in N(r)$ and any vertex $w \in N(v) \setminus N(r)$ the dart between v and w has orientation (v, w). Proceeding like this, we see that all darts have the correct orientation.

'$\Rightarrow$:' Let $\Gamma = (V, D, o)$ be an arborescence diverging from $r \in V$. Obviously, Γ does not contain any (directed) cycle because a (directed) cycle would lead to an (undirected) cycle in the underlying tree $U(\Gamma)$.

Assume $\operatorname{indeg}(r) > 0$. This means that there exists a dart d going from any vertex $v \neq r$ to r.

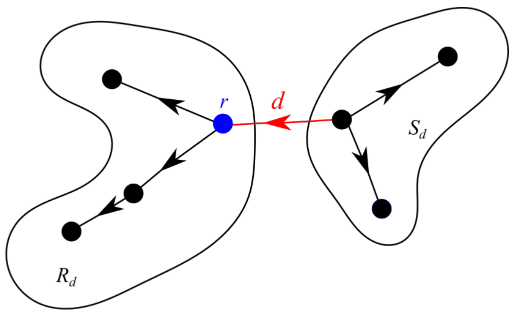


Figure 2.21: Dart d going from S_d to R_d.

If we consider the corresponding edge in the underlying tree $U(\Gamma)$ and look at the two connected components S_d and R_d of $U(\Gamma) - d$, we see that the dart d goes from S_d to R_d. This does not coincide with the construction of arborescences. Hence our assumption was wrong and $\operatorname{indeg}(r) = 0$.

It remains to show $\operatorname{indeg}(v) = 1$ for all $v \in V$ with $v \neq r$. Fix a vertex $v \in V \backslash \{r\}$. Let $k := \operatorname{indeg}(v)$ and $d_1, \ldots, d_k$ the incoming darts.

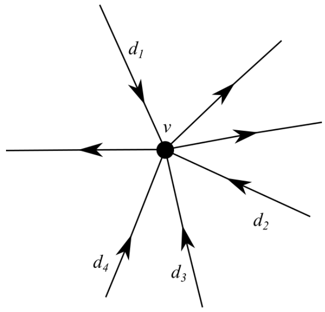


Figure 2.22: Vertex v with its four incoming darts $d_1, \ldots, d_4$.

To every d_j, $j \in [k]$, let e_j be the corresponding edge in the underlying tree $U(\Gamma)$. For $j \in [k]$, let T_j be the connected components of $U(\Gamma) - e_j$ (see Definition 2.1.12) with $v \notin T_j$. These components are pairwise disjoint. Otherwise there would be a cycle (see Figure 2.23 the red line together with the darts d_2 and d_3) in the tree $U(\Gamma)$.

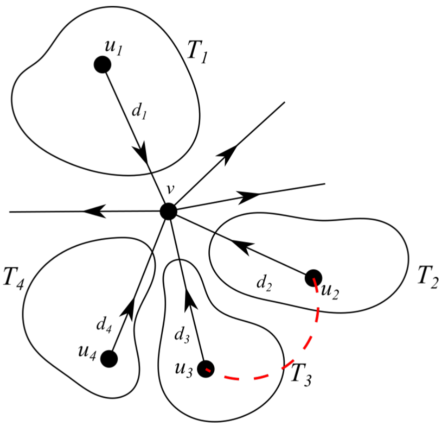


Figure 2.23: Vertex v with incident darts and the start-vertices u_i to every incoming dart d_i and the subtrees T_i.

For every incoming dart d_j to the vertex v, the start-vertex $u \neq v$ is in the subtree T_j. By the construction of arborescence, we know by the orientation of the dart d that our root r is in T_j aswell. Assume $\operatorname{indeg}(v) \geq 2$, then there are two darts d_i, d_j with $i, j \in [k]$ and $i \neq j$ such that the target-vertex $v = t(d_i) = t(d_j)$ is the same. The start-vertices are denoted by $u_i = s(d_i), u_j = s(d_j)$. Since we started with an arborescence, there are no multi-darts (see 2.4.4), so $u_i \neq u_j$. As we have just seen this implies $r \in T_j$ and $r \in T_i$, which is a contradiction to the pairwise disjointness of the subtrees T_j and T_i. This gives us $\operatorname{indeg}(v) \leq 1$ for all $v \in V \backslash \{r\}$.

If there was a vertex $\tilde{v} \in V \backslash \{r\}$ with $\operatorname{indeg}(\tilde{v}) = 0$, we would get

$$|D| \overset{2.3.6}{=} \sum_{v \in V} \operatorname{indeg}(v) = \underbrace{\operatorname{indeg}(\tilde{v}) + \operatorname{indeg}(r)}_{=0} + \sum_{v \in V\{r,\tilde{v}\}} \underbrace{\operatorname{indeg}(v)}_{\leq 1} \leq |V| - 2.$$

This is a contradiction to the connectivity of the underlying tree $U(\Gamma)$ by Proposition 2.2.7 (a). Therefore every from r distinct vertex v has incoming degree 1. □

Analogue to spanning trees (2.2.5), we define spanning arborescences.

2.4.7 Definition. A *spanning arborescence A diverging from r* of a digraph Γ is a subdigraph of Γ which is an arborescence diverging from r such that $V_A = V_\Gamma$.

2.4.8 Proposition. *Let G be a graph and $r \in V_G$ any vertex of G. There is a bijection between the set of spanning trees of G and the set of spanning arborescences diverging from r of the equivalent digraph $\widetilde{G}$ of G.*

Proof. Let T be a spanning tree of G . By construction 2.4.1 of an arborescence there is a unique arborescence diverging from r. This is a spanning arborescence of the equivalent digraph.

For the other direction, we consider any spanning arborescence of the equivalent digraph diverging from r. The underlying graph of the spanning arborescence is a tree using all vertices of Γ and with that all vertices of G. Hence it is a spanning tree of G. Both are inverse to each other. □

Chapter 3

Matrix-Tree Theorem

The aim of this chapter is to formulate and prove the Matrix-Tree Theorem for weighted (undirected) graphs. A special case of the Matrix-Tree Theorem was first proven by Rényi [CK78, Introduction]. The basis for this was already created in the last chapter by introducing the necessary terms in graph theory. It is hardly possible to find a direct proof of the undirected version. So we need to prove the Matrix-Tree Theorem first for weighted digraphs and use this theorem to easily conclude the version for undirected graphs. We use this theorem in chapter four in order to go back to the topic of hyperbolic polynomials and their hyperbolicity cones and show the main result of this thesis. Using the Matrix-Tree Theorem it is easy to calculate the number of spanning trees of an arbitrary graph just by calculating a determinant of the Laplacian Matrix (3.2.1). So Cayley's Formula is a corollary of the Matrix-Tree Theorem (3.1.11). Similar this works for digraphs such that we are able to calculate the number of spanning arborescences of a given digraph. For the whole chapter, we fix an integer $n \in \mathbb{N}$.

3.1 Matrix-Tree Theorem for digraphs

The Matrix-Tree Theorem is a statement about the connection of the spanning tree polynomial and the determinant of a matrix. The rule is sometimes also called Maxwell's or Kirchhoff's rule [CK78].

We are going to prove a version for weighted digraphs. So for a given digraph $\Gamma = (V, D, o)$, we assign a variable to every dart $d \in D$. It is also possible to work with dart-weights, which is a function

$$\begin{aligned} \omega : D &\to \mathbb{R}\left[(X_d)_{d\in D}\right], \\ d &\mapsto X_d. \end{aligned}$$

3.1.1 Definition. For any digraph $\Gamma = (V, D, o)$ with a fixed numeration of the vertices $V = \{v_1, \ldots, v_n\}$, we define the *weighted Laplacian* $L_\Gamma = (l_{ij})_{1\le i,j\le n}$ as the $n \times n$-matrix defined as

$$l_{ij} := \begin{cases} -\sum\limits_{\substack{d\in D,\\ o(d)=(v_j,v_i)}} X_d, & \text{if } i \neq j; \\ \sum\limits_{\substack{d\in D,\\ t(d)=v_i}} X_d, & \text{if } i = j. \end{cases}$$

Remember that $t(d)$ denotes the target-vertex of the dart d, 2.3.1. In the literature, this matrix is sometimes also called Kirchhoff Matrix.

3.1.2 Remark. Since we do not consider digraphs with loops (i.e. a dart with same start- and end-vertex), the sum for the diagonal-entries l_{ii} for all $i \in [n]$ goes over all darts $d \in D$ with $t(d) = v_i$ but $s(d) \neq v_i$.

Although, we do not consider digraphs with loops, it would be possible. A loop would not change the weighted Laplacian in case, we add $s(d) \neq v_i$ to the condition of the sum for the diagonal entries.

A very important and obvious property of the weighted Laplacian is the vanishing row-sum.

3.1.3 Remark. The i-th row sum $\sum\limits_{j\in[n]} l_{ij}$ vanishes because

$$\begin{aligned}
\sum_{j=[n]} l_{ij} &= \sum_{j\in[n]\setminus\{i\}} l_{ij} + l_{ii} \\
&= \sum_{j\in[n]\setminus\{i\}} \left(- \sum_{\substack{d\in D \\ o(d)=(v_j,v_i)}} X_d \right) + \sum_{\substack{d\in D \\ t(d)=v_i}} X_d \\
&= - \sum_{j\in[n]\setminus\{i\}} \sum_{\substack{d\in D \\ o(d)=(v_j,v_i)}} X_d + \sum_{j\in[n]\setminus\{i\}} \sum_{\substack{d\in D \\ o(d)=(v_j,v_i)}} X_d \\
&= 0
\end{aligned}$$

for every $i \in [n]$. So the columns of L_Γ are linear dependent and the determinant of the Laplacian L_Γ is zero if there is at least one vertex, too.

3.1.4 Example. Consider the digraph Γ on three vertices (see Figure 3.1)

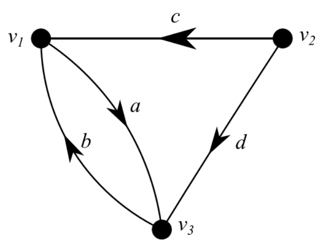


Figure 3.1: Digraph Γ on the vertices $V = \{v_1, v_2, v_3\}$ with four darts.

The weighted Laplacian of the digraph Γ drawn in Figure 3.1 is

$$L_\Gamma = \begin{pmatrix} X_b + X_c & -X_c & -X_b \\ 0 & 0 & 0 \\ -X_a & -X_d & X_a + X_d \end{pmatrix}. \tag{3.1}$$

3.1.5 Definition.

(a) Let Γ be a digraph and $r \in V_\Gamma$ a fixed vertex. By $\mathcal{A}_{\Gamma,r}$, we denote the set of all spanning arborescences of Γ diverging from r.

(b) The *spanning arborescence polynomial* P_Γ is defined as

$$P_\Gamma := \sum_{A\in\mathcal{A}_{\Gamma,r}} \prod_{d\in D_A} X_d.$$

3.1.6 Remark. For every arborescence A, there is only one spanning arborescence of A namely A itself. So $P_A = \prod_{d\in D_A} X_d$.

3.1.7 Definition. For $n \in \mathbb{N}$, any matrix $A \in \mathrm{M}_n(R)$ over a commutative ring R, we denote the $(n-1)\times(n-1)$ matrix obtained by deleting out the j-th column and j-th row of A as A_j for any $j \in [n]$.

For the Laplacian L_Γ of a digraph Γ with enumerated vertices $v_1, \dots, v_n$ and a fixed vertex $r \in V_\Gamma$, $(L_\Gamma)_r$ describes the matrix L_Γ with deleted j-th column and row if $r = v_j$.

Now, we are able to formulate the main theorem of this section, the Matrix-Tree Theorem for digraphs.

3.1.8 Theorem (Matrix-Tree Theorem for digraphs, 1^{st} version)**.** *Let $\Gamma = (V, D, o)$ be a digraph with a vertex r. The following equality holds:*

$$\det(L_\Gamma)_r = P_\Gamma = \sum_{A\in\mathcal{A}_{\Gamma,r}} P_A = \sum_{A\in\mathcal{A}_{\Gamma,r}} \prod_{d\in D_A} X_d. \tag{3.2}$$

Before we prove the Matrix-Tree Theorem, we will look at some examples and prove a corollary.

3.1.9 Example. Let us consider the digraph drawn in Figure 3.1. For $r = v_3$, we get with the Matrix-Tree Theorem

$$P_\Gamma = \det(L_\Gamma)_r = \begin{vmatrix} X_b + X_c & -X_c \\ 0 & 0 \end{vmatrix} = 0. \tag{3.3}$$

This means, there are no spanning arborescences because the sum appearing in the spanning tree polynomial must be empty. There is no other possibility for the sum to vanish. Considering the graph, we can verify that there are no spanning arborescences of the graph Γ diverging from v_3.

On the other hand, considering the vertex v_2 as root for the spanning arborescences (i.e. $r = v_2$), there are the following spanning arborescences.

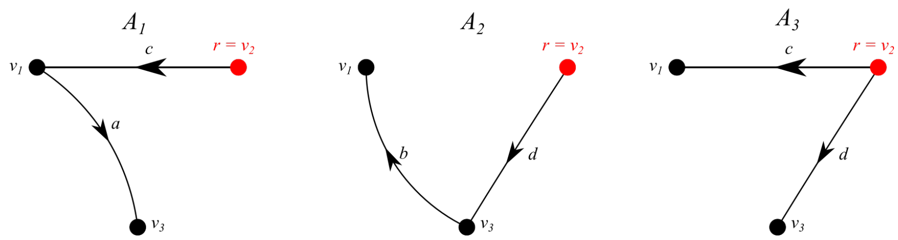


Figure 3.2: The three spanning arborescences A_1, A_2, A_3 of Γ (for Γ see Figure 3.1) diverging from $r = v_2$.

The Matrix-Tree Theorem says that the spanning arborescence polynomial is given as

$$\begin{aligned} P_\Gamma = \det(L_\Gamma)_2 = \begin{vmatrix} X_b + X_c & -X_b \\ -X_a & X_a + X_d \end{vmatrix} &= (X_b + X_c)(X_a + X_d) - X_a X_b \\ &= X_a X_c + X_c X_d + X_b X_d. \end{aligned}$$

It is not by chance that the number of summands of the determinant coincides with the number of spanning trees. In the following corollary we will prove this.

Furthermore, from the spanning arborescence polynomial we can conclude to the spanning arborescences of the considered graph. Every summand stands for one spanning arborescence. The variables show which edges appear in the corresponding spanning tree.

3.1.10 Corollary. *Let $\Gamma = (V, D, o)$ be a digraph on n vertices $v_1, v_2, \ldots, v_n$. Furthermore, fix a vertex r and denote by N_Γ the number of spanning arborescences of Γ diverging from r. Then $N_\Gamma = \det(L_\Gamma)_r((1)_{d\in D})$. For the Complete Digraph Γ (all possible darts in the digraph without multi-darts, see 2.3.4), we get $N_\Gamma = n^{n-2}$.*

Proof. We evaluate the polynomial at 1 for every variable X_d with $d \in D$ in the spanning tree polynomial. We write $\mathbf{1} = (1)_{d\in D}$ to make it easier to read. Then $(P_A)(\mathbf{1}) = \left(\prod_{d\in D_A} X_d\right)(\mathbf{1}) = 1$ for every arborescence $A \in \mathcal{A}_\Gamma$ diverging from r, so

$$N_\Gamma = \sum_{A\in\mathcal{A}_{\Gamma,r}} 1 = P_\Gamma(\mathbf{1}) \overset{3.1.8}{=} \det\left(L_\Gamma\right)_r(\mathbf{1}).$$

The Complete Digraph Γ has the Laplacian

$$(L_\Gamma)(\mathbf{1}) = \begin{pmatrix} n-1 & & -1 \\ & \ddots & \\ -1 & & n-1 \end{pmatrix} \in \mathrm{M}_n(\mathbb{R})$$

evaluated at $\mathbf{1}$. The determinant of $(L_\Gamma)_r(\mathbf{1})$ is the characteristic polynomial of the matrix $A \in \mathrm{M}_{n-1}(\mathbb{R})$ with every entry $a_{ij} = 1$ for $i, j \in [n-1]$, evaluated at n. So $\det(L_\Gamma)_r = n^{n-2}$. $\square$

3.1.11 Remark. This is another proof for Cayley's Formula (see Theorem 2.2.8) using the bijection between spanning trees of a graph and the spanning arborescences diverging from a fixed vertex of the equivalent digraph 2.4.8.

If there is a vertex v_i without (directed) path going from r to v_i, then there is no spanning arborescence, hence $\det(L_\Gamma)_r = 0$.

Proof of the Matrix-Tree Theorem, [Tut84]. WLOG, we assume that $r = v_n$. This is possible because the determinant only changes the sign if we swap two rows or columns. We need to swap two rows and two columns, so the sign stays the same.

First we study the case $n = 1$: The determinant of a matrix of size 0×0, is 1. On the other hand the graph with only one vertex, has one spanning arborescence diverging from this vertex, the graph itself, but the product is empty since there are no darts. So the right hand side of equation (3.2) is 1, too. Hence the case $n = 1$ is shown.

Now, let $n > 1$. The determinant of $(L_G)_r = (l_{ij})_{i,j\in[n-1]}$ is easy to calculate with the Leibniz-Formula:

$$\det(L_G)_r = \sum_{\tau\in S_{n-1}} \operatorname{sgn}(\tau) \prod_{i=1}^{n-1} l_{i\tau(i)}.$$

We call the product $p_\tau := \prod_{i=1}^{n-1} l_{i\tau(i)}$ the *initial product* of τ. We can split the initial product into two products, $p_{\tau,1}$ and $p_{\tau,2}$, where $p_{\tau,1}$ is the product consisting of the fixpoints of τ and $p_{\tau,2}$ are the remaining factors, so

$$p_{\tau,1} = \prod_{\substack{i=1,\\ \tau(i)=i}} l_{ii} \quad \text{and}$$
$$p_{\tau,2} = \prod_{\substack{i=1,\\ \tau(i)\neq i}} l_{i\tau(i)}.$$

If τ consists not only out of fixpoints, we can split the factors appearing in $p_{\tau,2}$ into cycles. This means if $\tau = (a_1\ \tau(a_1)\ \cdots\ \tau^{l_1-1}(a_1))\cdots(a_k\ \tau(a_k)\ \cdots\ \tau^{l_k-1}(a_k))$, where $l_1,\ldots,l_k \in \mathbb{N}$ are the length of the cycles and $k \in \mathbb{N}$ the number of cycles. This leads to the product

$$\prod_{i=1}^{k} \left(\prod_{j=0}^{l_i-1} l_{\tau^j(a_i)\tau^{j+1}(a_i)} \right).$$

Let $n_0(\tau)$ denote the number of even cycles in τ. Then $\operatorname{sgn}(\tau) = (-1)^{n_0(\tau)}$ for every $\tau \in S_{n-1}$. With the definition of $(L_\Gamma)_r$, we see that every p_τ leads to one or more summands of the form $\prod_{d\in D_\Delta} X_d$ for a spanning subdigraph Δ. So it is possible to write

$$\det(L_\Gamma)_r = \sum_{\Delta} N(\Delta) \prod_{d\in D_\Delta} X_d, \tag{3.4}$$

where the sum goes over all spanning subdigraphs Δ of Γ and $N(\Delta) \in \mathbb{Z}$ for every spanning subdigraph Δ. It remains to determine the integers $N(\Delta)$.

If $N(\Delta) \neq 0$ for a spanning subdigraph Δ, the product $\prod_{d\in D_\Delta} X_d$ appears in at least one in p_τ, $\tau \in S_{n-1}$. Since the n-th row of L_G is deleted in $(L_G)_r$, l_{ni} does not appear in any initial product. Hence X_d with $d = (v_i, v_n)$, $i \in [n-1]$ does not appear in any initial product p_τ, $\tau \in S_{n-1}$ and the incoming-degree $\operatorname{indeg}_\Delta(v_n) = 0$ must vanish for all spanning subdigraphs Δ with $N(\Delta) \neq 0$. Furthermore, for every $i \in [n-1]$ there is exactly one $j \in [n-1]\backslash\{i\}$ such that X_d with $d = (v_j, v_i)$ appears in the initial product. To verify this, note that the diagonal elements are constructed such that the row sum vanishes. This means in the i-th diagonal element, X_d appears only if d is a dart ending in v_i. So all subdigraphs Δ with $N(\Delta) \neq 0$ have $\operatorname{indeg}_\Delta(v_n) = 0$ and $\operatorname{indeg}_\Delta(v_i) = 1$ for all $i \in [n-1]$. So we only consider those subdigraphs in the sum in (3.4).

Let p_{id} be the initial product belonging to the permutation $\mathrm{id} \in S_{n-1}$ with $n-1$ fixpoints. So

$$p_{\mathrm{id}} = \prod_{i=1}^{n-1} l_{ii} = \prod_{i=1}^{n-1} \left(- \sum_{\substack{j=1,\\ j\neq i}} l_{ij} \right) = \prod_{i=1}^{n-1} \sum_{\substack{j=1,\\ j\neq i}} \sum_{\substack{d\in D\\ o(d)=(v_j,v_i)}} X_d.$$

For every spanning arborescence $A \in \mathcal{A}_{\Gamma,r}$ the product $\prod_{d\in D_A} X_d$ is a summand of p_{id} and every such summand belongs to an arborescence. This contribute ! to $N(A)$ for every spanning arborescence $A \in \mathcal{A}_{\Gamma,r}$.

Assume $N(A) > 1$ for a spanning arborescence A. Then the product comes also from another initial product p_τ, $\tau \in S_{n-1}\backslash\{\mathrm{id}\}$. Then τ has at least one cycle of length > 1. This cycle corresponds to a cycle in the spanning arborescence.

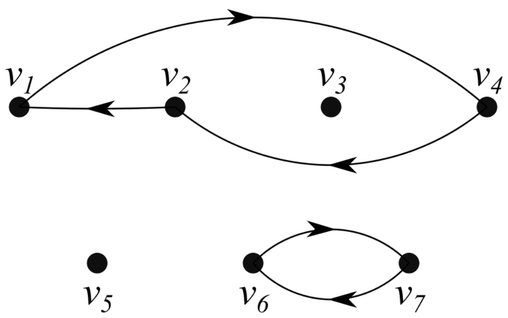


Figure 3.3: Cycles of a subdigraph belonging to the permutation $\tau = (1\,2\,4)(67)$.

This is a contradiction to the definition of a spanning arborescence. So $N(A) = 1$ for all spanning arborescences $A \in \mathcal{A}_{\Gamma,r}$.

Now we need to study the spanning digraphs Δ of Γ with $\mathrm{indeg}_\Delta(v_n) = 0$, $\mathrm{indeg}_\Delta(v_i) = 1$ for all $i \in [n-1]$ but which are no spanning arborescences of Γ. Lemma 2.4.6 implies that Δ must contain a cycle. So the product $\prod_{d\in\Delta} X_d$ comes from p_τ if and only if every cycle of τ corresponds to a cycle in Δ. The cycles in Δ are vertex disjoint. So assume in Δ, there are $k > 0$ distinct tours (2.3.10) with disjoint vertex sets. Let $\mathscr{T}$ be the set of these k tours. So $\prod_{d\in D_\Delta} X_d$ is in the initial product p_τ if and only if the cycles in τ are a subset of $\mathscr{T}$. In particular, it is also possible to have $\tau = \mathrm{id}$. Assume τ has $j \in \{0,\dots,k\}$ cycles. Since the non-diagonal entries of $(L_G)_r$ have a negative sign, the initial product p_τ contributes $(-1)^{n_0(\tau)}(-1)^{j-n_0(\tau)} = (-1)^j$ to $N(\Delta)$. The first part $(-1)^{n_0(\tau)}$ comes from the Leibniz-Formula and the second part from the minus sign of the non-diagonal entries. There are $\binom{k}{j}$ possibilities to choose exactly j of those k cycles and there is always exactly one initial product corresponding to this. So

$$N(\Delta) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j = (1-1)^k = 0.$$

So in equation (3.4), the only non-vanishing summands are the ones belonging to an arborescence and in the case of an arborescence A the integer is $N(A) = 1$. So the statement is proven. □

3.2 Matrix-Tree Theorem for (undirected) graphs

In the last section, we proved the Matrix-Tree Theorem for digraphs. In this section, we are going to formulate and prove the undirected version of the Matrix-Tree Theorem. This, undirected version, is the statement we are going to use in the last chapter in order to prove the main result of this thesis. Similar as in the directed case, we assign to each edge $e \in E$ an edge variable X_e.

3.2.1 Definition. We define the weighted Laplacian $L_G = (l_{ij})_{1\le i,j\le n}$ for a graph $G = (V, E, \epsilon)$ on the vertices $v_1, v_2, \ldots, v_n$ by

$$l_{ij} = \begin{cases} -\sum\limits_{\substack{e\in E,\\ \epsilon(e)=\{v_i,v_j\}}} X_e, & \text{if } i \neq j; \\ \sum\limits_{\substack{e\in E,\\ v_i\in\epsilon(e)}} X_e, & \text{if } i = j. \end{cases} \tag{3.5}$$

3.2.2 Remark. Of course, it is also possible to work with edge-weights instead of the assigned edge-variables.

3.2.3 Proposition. *Let $G = (V, E, \epsilon)$ be a graph with vertices $v_1, \ldots, v_n$. The weighted Laplacian is*

$$L_G = \sum_{e\in E} X_e(\mathbf{e}_{v_e} - \mathbf{e}_{w_e})(\mathbf{e}_{v_e} - \mathbf{e}_{w_e})^T,$$

where $\epsilon(e) = \{v_e, w_e\}$ are the incident vertices. The vectors $(\mathbf{e}_k)_{k\in[n]}$ denote the standard-basis vectors of $\mathbb{R}^n$ and for all $e \in E$ and $j \in [n]$ it is $\mathbf{e}_{v_e} = \mathbf{e}_j$ if and only if $v_e = v_j$.

Proof. The matrix $A_e := (\mathbf{e}_{v_e} - \mathbf{e}_{w_e})(\mathbf{e}_{v_e} - \mathbf{e}_{w_e})^T$ has two times an one as a diagonal entry and all other diagonal entries are zero for all $e \in E$. If v_i and v_j, $i, j \in [n]$ are the two incident vertices to a fixed edge e, the matrix $A_e = (a_{lk})_{l,k\in[n]}$ has $a_{ii} = a_{jj} = 1$ and $a_{ij} = a_{ji} = -1$ and all other entries are zero.

For every edge $e \in E$ the edge-variable X_e counts one to the diagonal elements l_{ii} and l_{jj} if e is an edge going from v_i to v_j. So the diagonal elements l_{jj} is the sum of all to v_j incident edges. This is also the case in the matrix $\sum\limits_{e\in E} X_e(\mathbf{e}_{v_e} - \mathbf{e}_{w_e})(\mathbf{e}_{v_e} - \mathbf{e}_{w_e})^T$. Furthermore, the edge e with $\epsilon(e) = \{v_i, v_j\}$ contributes the summand $-X_e$ to the matrix-entries l_{ij} and l_{ji} for $i \neq j$. Since the ij-th entry of A_e is -1, the edge e contributes also $-X_e$ in the matrix $\sum\limits_{e\in E} X_e(\mathbf{e}_{v_e} - \mathbf{e}_{w_e})(\mathbf{e}_{v_e} - \mathbf{e}_{w_e})^T$ to the ij-th and ji-th entry. So all entries of the two matrices coincide. □

3.2.4 Remark.

(a) Since the edges have no direction, the weighted Laplacian of a graph is symmetric.

(b) Analogue to the digraphs (see Remark 3.1.3), the row sum is zero and so the determinant of the weighted Laplacian vanishes.

3.2.5 Definition. Let $\mathcal{T}_G$ be the set of all spanning trees of a graph G. The *spanning tree polynomial* is defined as

$$T_G := \sum_{T \in \mathcal{T}_G} \prod_{e \in E_T} X_e.$$

3.2.6 Theorem (Matrix-Tree Theorem for graphs)**.** *Let $G = (V, E, \epsilon)$ be a graph on the vertices $v_1, v_2, \ldots, v_n$ (with a fixed numeration of the vertices), and one of the vertices is supposed to be r. Then*

$$\det(L_G)_r = T_G.$$

Proof. To the undirected graph G, we consider the equivalent digraph $\widetilde{G} = (V, D, o)$, defined in 2.3.12. To every dart $d \in D$, $d = e^+$ or $d = e^-$ for any edge $e \in E$, we assign the edge-variable X_e.

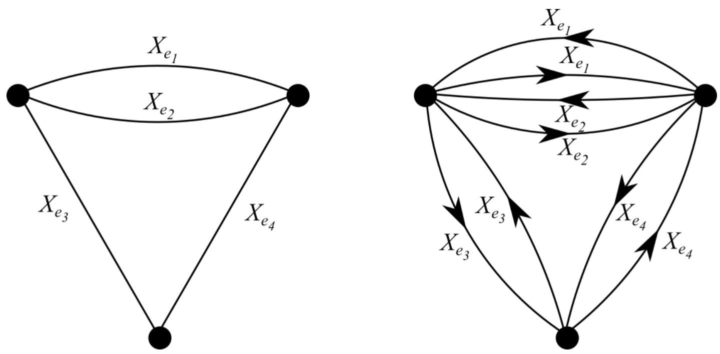


Figure 3.4: On the left-hand side, we see a graph G with its equivalent digraph $\widetilde{G}$ on the right-hand side. In both, the graph and the digraph, the edges/ darts are labelled with its edge-weights/ dart-weights.

The Laplacian of the equivalent digraph $\widetilde{G}$ coincides with the Laplacian of the considered graph G and so

$$\det(L_G)_r = \det(L_{\widetilde{G}})_r \overset{3.1.8}{=} \sum_{A \in \mathcal{A}_{\Gamma_r}} \prod_{d \in D_A} X_d = \sum_{T \in \mathcal{T}_G} \prod_{e \in E_T} X_e,$$

for a fixed vertex $r \in V$. For the last equality, we used the bijection between spanning trees and spanning arborescences diverging from r mentioned in Proposition 2.4.8. The product remains the same for every spanning tree/ spanning arborescence because we use for any edge e the dart e^+ or e^- and they have the same edge-weights. □

3.2.7 Remark. The spanning-tree polynomials are independent of the chosen vertex r and so the determinant is independent of the deleted row and column.

3.3 Hyperbolicity cones of graphs

Back to the theory of hyperbolicity cones. We are able to apply the Matrix-Tree Theorem presented above (Theorem 3.2.6) to the hyperbolicity cones of some spanning tree polynomials in order to see that the hyperbolicity cones are spectrahedral.

3.3.1 Definition. The *hyperbolicity cone of a connected graph G* is the hyperbolicity cone of the corresponding spanning tree polynomial T_G.

In the case we consider a disconnected graph G, there are no spanning trees (see 2.2.6) such that $T_G = 0$ (in $\mathbb{R}[\mathbf{X}]$). This is a non-hyperbolic polynomial. In the connected case, we show:

3.3.2 Proposition. *[Brä13, p.3]. The hyperbolicity cone of any connected graph is spectrahedral.*

Proof. We consider a connected graph G on n vertices and fix an arbitrary $i \in [n]$. In Proposition 3.2.3, we have seen that the Laplacian of G is

$$L_G = \sum_{e \in E} X_e (\mathbf{e}_{v_e} - \mathbf{e}_{w_e})(\mathbf{e}_{v_e} - \mathbf{e}_{w_e})^T.$$

The matrices $A_e := (\mathbf{e}_{v_e} - \mathbf{e}_{w_e})(\mathbf{e}_{v_e} - \mathbf{e}_{w_e})^T$ for all $e \in E$ are positive semi-definite and they stay positive semi-definite after deleting the i-th row and the i-th column. So the sum over all those matrices $(A_e)_i$ is positive semi-definite as well and is the same as $\det(L_G)_i(\mathbf{1})$, where $\mathbf{1} = (1)_{e \in E}$. Together with the Matrix-Tree Theorem 3.2.6, we see this is equal to the number of spanning trees of G. So $\det(L_G)_i(\mathbf{1})$ is a positive integer because G is connected and so there is at least one spanning tree (2.2.6). This implies that $(L_G)_i(\mathbf{1})$ is positive definite.

As we have already shown in the proof of Proposition 1.2.11 this implies that T_G is hyperbolic in direction $\mathbf{1}$ and

$$\overline{\Lambda}(T_G, \mathbf{1}) = \{\mathbf{x} \in \mathbb{R}^n : \ (L_G)_i(\mathbf{x}) \succeq 0\}.$$

□

Chapter 4

Hyperbolicity Cones of Elementary Symmetric Polynomials are Spectrahedral

In this chapter, we want to prove that hyperbolicity cones of elementary symmetric polynomials are spectrahedral. This is the main theorem of this thesis. We present the proof of Brändén [Brä13]. In the last section of chapter three, we have already seen that the hyperbolicity cone of any connected graph is spectrahedral. So the idea of this proof is to construct a connected graph such that the corresponding spanning tree polynomial contains an elementary symmetric polynomial as a factor. To determine the spanning tree polynomial belonging to the graph, we use the Matrix-Tree Theorem 3.2.6 presented in the last chapter. In this way, we want to define a graph such that the hyperbolicity cone of the spanning tree polynomial is spectrahedral. This is sufficient for the hyperbolicity cone of the elementary symmetric polynomial to be spectrahedral if additionally a subset property of the hyperbolicity cones holds. This statement is verified by the next theorem. For the whole chapter, we fix an integer $n \in \mathbb{N}$.

4.0.1 Theorem. *Let $p \in \mathbb{R}[\mathbf{X}] = \mathbb{R}[X_1, \dots, X_n]$ be a homogeneous polynomial, hyperbolic in direction $\mathbf{d} \in \mathbb{R}^n$. If there are finitely many symmetric matrices $A_1, \dots, A_n \in Sym_m(\mathbb{R})$, $m \in \mathbb{N}$, such that $\sum_{i=1}^{n} d_i A_i \succ 0$ and if additionally there is a homogeneous polynomial $q \in \mathbb{R}[\mathbf{X}]$ such that*

(1) $p \cdot q = \det\left(\sum_{i=1}^{n} X_i A_i\right)$ and

(2) $\overline{\Lambda}(p, \mathbf{d}) \subseteq \overline{\Lambda}(q, \mathbf{d})$,

the hyperbolicity cone $\overline{\Lambda}(p, \mathbf{d})$ is spectrahedral.

Proof. Let $A_1, \dots, A_n$ be symmetric matrices and q a homogeneous polynomial as mentioned in the assumptions of the theorem. We need to show that $\overline{\Lambda}(p, \mathbf{d})$ is spectrahedral. The polynomial $p \cdot q$ is hyperbolic in direction $\mathbf{d}$ because of assumption (1). This we have already proven in the proof of 1.2.11. Furthermore, Proposition 1.2.11 also shows that the hyperbolicity cone $\overline{\Lambda}(p \cdot q, \mathbf{d})$ is the spectrahedral cone

$$\left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i A_i \succeq 0 \right\}.$$

With assumption (2) it follows

$$\Lambda(p \cdot q, \mathbf{d}) \overset{1.3.1}{=} \Lambda(p, \mathbf{d}) \cap \Lambda(q, \mathbf{d}) \overset{(2)}{=} \Lambda(p, \mathbf{d}).$$

This shows $\overline{\Lambda}(p, \mathbf{d}) = \overline{\Lambda}(p \cdot q, \mathbf{d})$ is spectrahedral, too. ☐

4.0.2 Remark. In the theorem above, we did not assume that the polynomial q is hyperbolic but only that it is homogeneous. The hyperbolicity follows directly out of Lemma 1.3.1 and Proposition 1.2.11.

This theorem is the basic idea of the proof of Theorem 4.4.1. It is a sufficient condition for a hyperbolicity cone to be spectrahedral and it is possible to apply this theorem to the hyperbolicity cone of an elementary symmetric polynomial. We need a homogeneous polynomial, such that one of its factors is an elementary symmetric polynomial and the polynomial itself must have a determinant-presentation as in assumption (1) of Theorem 4.0.1. However, we cannot forget about the other factors. The other factors need to fulfil the subset condition of the corresponding hyperbolicity cones (see Theorem 4.0.1 (2)). With a closer look at condition (1) and the Matrix-Tree Theorem 3.2.6, the crucial idea is given. The Laplacian of a graph has the form $\sum\limits_{k=1}^{n} X_k A_k$ for certain matrices (see Proposition 3.2.3). Hence, we need a graph such that the spanning tree polynomial contains an elementary symmetric polynomial as factor and fulfils all other conditions of the theorem above.

4.1 Elementary symmetric polynomials

Before we start with the recursive construction of the graphs, we first need to show some technical properties of the elementary symmetric polynomials. These properties are necessary for the recursive construction and the computation of the spanning tree polynomial $H_{k,k}$. We start with a recurrence relation of the elementary symmetric polynomials.

4.1.1 Lemma. *Let $S \subseteq [n]$ and $k \in \mathbb{N}$. For any $j \in S$ the following equation holds:*

$$\sigma_k(S) = \sigma_k(S\backslash\{j\}) + X_j \sigma_{k-1}(S\backslash\{j\}).$$

Proof. By definition of the elementary symmetric polynomials (1.3.7), $\sigma_k(S)$ consists of a sum over all subsets of S with cardinality k. Now fix some $j \in S$. We divide the set of subsets of S with cardinality k in those including j and those without j. Let $\mathscr{S}_1 := \{T \subseteq S : |T| = k \wedge j \notin T\}$ and $\mathscr{S}_2 := \{T \subseteq S : |T| = k \wedge j \in T\}$. So for the elementary symmetric polynomial $\sigma_k(S)$, we get

$$\sigma_k(S) = \sum_{\substack{T \subseteq S \\ |T|=k}} \prod_{i \in T} X_i = \sum_{T \in \mathscr{S}_1} \prod_{i \in T} X_i + \sum_{T \in \mathscr{S}_2} \prod_{i \in T} X_i. \tag{4.1}$$

In the first sum the variable X_j does not appear, but it is still a homogeneous polynomial of degree k in the variables $(X_i)_{i \in S\backslash\{j\}}$. Hence

$$\sum_{T \in \mathscr{S}_1} \prod_{i \in T} X_i = \sum_{\substack{T \subseteq S\backslash\{j\}, \\ |T|=k}} \prod_{j \in T} X_j = \sigma_k(S\backslash\{j\}).$$

Obviously, the variable X_j appears in the second sum in the equation (4.1) in each summand exactly once. Hence we can factor it out. After factoring it out, it remains a homogeneous polynomial of degree $k-1$ in the variables $(X_i)_{i\in S\backslash\{j\}}$, which is the $(k-1)$-th elementary symmetric polynomial in $(X_i)_{i\in S\backslash\{j\}}$. It is

$$\sum_{T\in\mathscr{S}_2}\prod_{i\in T}X_i = X_j\cdot\sum_{T\in\mathscr{S}_2}\prod_{i\in T\backslash\{j\}}X_i = X_j\cdot\sum_{\substack{T\subseteq S\backslash\{j\}\\|T|=k-1}}\prod_{i\in T}X_i = X_j\cdot\sigma_{k-1}(S\backslash\{j\}).$$

If we put the two sums together in the first equation (4.1), we get the claim. □

It is also possible to determine σ_k only depending on σ_{k-1}.

4.1.2 Lemma. *Let $S\subseteq[n]$ denote a set of positive integers and $k\in\mathbb{N}$ a natural number. We have*

$$k\sigma_k(S) = \sum_{j\in S}X_j\sigma_{k-1}(S\backslash\{j\}).$$

Proof. We use the definition of the elementary symmetric polynomial $\sigma_k(S)$ and with some transformations, we get:

$$\begin{aligned}
k\sigma_k(S) &= k\sum_{\substack{T\subseteq S\\|T|=k}}\prod_{i\in T}X_i\\
&= \sum_{\substack{T\subseteq S\\|T|=k}}\left(\sum_{j\in T}\prod_{i\in T}X_i\right)\\
&= \sum_{\substack{T\subseteq S\\|T|=k}}\sum_{j\in T}X_j\prod_{i\in T\backslash\{j\}}X_i\\
&= \sum_{j\in S}\sum_{\substack{T\subseteq S,\\|T|=k,\\j\in T}}X_j\prod_{i\in T\backslash\{j\}}X_i\\
&= \sum_{j\in S}X_j\sum_{\substack{T\subseteq S\backslash\{j\},\\|T|=k-1}}\prod_{i\in T}X_i\\
&= \sum_{j\in S}X_j\sigma_{k-1}(S\backslash\{j\}).
\end{aligned}$$

□

4.1.3 Definition. For a subset $S\subseteq[n]$ and a polynomial $p\in\mathbb{R}[\mathbf{X}]$, we define the derivative

$$\partial^S p := \left(\prod_{i\in S}\frac{\partial}{\partial X_i}\right)p.$$

This is a repeated partial derivative (see Definition 1.3.2).

4.1.4 Remark. The definition of ∂^S is well-defined, because it does not matter in which order we form the partial derivative.

We want to determine this kind of repeated partial derivative of the elementary symmetric polynomials.

4.1.5 Lemma. *Let $S \subseteq [n]$ and $k \in \mathbb{N}$. Then*

$$\partial^S \sigma_k = \begin{cases} \sigma_{k-|S|}([n]\backslash S), & \text{if } k \geq |S|; \\ 0, & \text{if } k < |S|. \end{cases}$$

Proof. The statement follows directly from

$$\frac{\partial}{\partial X_j}\sigma_l(S) = \frac{\partial}{\partial X_j}\left(\sum_{\substack{T \subseteq S \\ |T|=l}} \prod_{i \in T} X_i\right) = \sum_{\substack{T \subseteq S\backslash\{j\} \\ |T|=l-1}} \prod_{i \in T} X_i = \sigma_{l-1}(S\backslash\{j\})$$

if $l \geq 1$. If $l = 0$, the elementary symmetric polynomial $\sigma_0(S) = 1$. So the partial derivative vanishes. □

4.2 Motivation

In this section, we want to prove that the second elementary symmetric polynomial has a spectrahedral hyperbolicity cone. This special case of the theorem shows how the idea of the proof for the theorem came up. Afterwards, we only need to expand this idea to higher dimensions. To show the case $k = 2$, we do not need the following parts in this generality. We define it in this way because we will need it later on for the proof and the transformation in Proposition 4.2.3 shows us the motivation of the general proof.

4.2.1 Definition. Let $S \subseteq [n]$ be any non-empty subset and $k \in \mathbb{N}$ such that $k \leq |S|$. We define

$$q_k(S) := \frac{\sigma_k(S)}{\sigma_{k-1}(S)}.$$

This is a rational function in $\mathbb{R}(\mathbf{X})$.

4.2.2 Lemma. *For any non-empty set $S \subseteq [n]$ and integer $k \in \mathbb{N}$ with $2 \leq k \leq |S|$ it holds*

$$kq_k(S) = \sum_{j \in S} \frac{X_j q_{k-1}(S\backslash\{j\})}{X_j + q_{k-1}(S\backslash\{j\})}.$$

Proof. We use the two previous lemmata 4.1.1 and 4.1.2. The first mentioned one, we apply to the denominator and the second one to the numerator in the definition of $q_k(S)$, such that we get

$$\begin{aligned}
kq_k(S) &= \frac{k\sigma_k(S)}{\sigma_{k-1}(S)} \\
&\overset{4.1.2}{=} \frac{1}{\sigma_{k-1}(S)} \cdot \sum_{j\in S} X_j \sigma_{k-1}(S\backslash\{j\}) \\
&\underset{k\geq 2}{\overset{4.1.1}{=}} \sum_{j\in S} \frac{X_j\sigma_{k-1}(S\backslash\{j\})}{\sigma_{k-1}(S\backslash\{j\}) + X_j\sigma_{k-2}(S\backslash\{j\})} \\
&= \sum_{j\in S} X_j \cdot \frac{\sigma_{k-1}(S\backslash\{j\})}{\sigma_{k-2}(S\backslash\{j\})} \cdot \frac{1}{X_j + \frac{\sigma_{k-1}(S\backslash\{j\})}{\sigma_{k-2}(S\backslash\{j\})}} \\
&= \sum_{j\in S} X_j q_{k-1}(S\backslash\{j\}) \cdot \frac{1}{X_j + q_{k-1}(S\backslash\{j\})}.
\end{aligned}$$

This is the claimed statement. □

4.2.3 Proposition. *We have*

$$2q_2([n]) \cdot \prod_{j=1}^{n} (X_j + \sigma_1([n]\backslash\{j\})) = 2\sigma_2\sigma_1^{n-1}.$$

Proof. We consider a linear variable transformation. So we study the variables $(Y_j)_{j\in[n]}$ such that

$$Y_j := q_1([n]\backslash\{j\}). \tag{4.2}$$

If we study Y_j, we see:

$$Y_j = q_1([n]\backslash j) = \frac{\sigma_1([n]\backslash\{j\})}{\sigma_0([n]\backslash\{j\})} = \sigma_1([n]\backslash\{j\}). \tag{4.3}$$

Let us rewrite the term $2q_2([n])$ as

$$\begin{aligned}
2q_2([n]) \overset{4.2.2}{=} \sum_{j=1}^{n} \frac{X_j \cdot q_1([n]\backslash\{j\})}{X_j + q_1([n]\backslash\{j\})} &\overset{(4.2)}{=} \sum_{j=1}^{n} \frac{X_j \cdot Y_j}{X_j + Y_j} \\
&\overset{(4.3)}{=} \sum_{j=1}^{n} \frac{X_j \cdot \sigma_1([n]\backslash\{j\})}{X_j + \sigma_1([n]\backslash\{j\})}.
\end{aligned} \tag{4.4}$$

The factors $X_j + \sigma_1([n]\backslash\{j\})$ of the product mentioned in the lemma are

$$X_j + \sigma_1([n]\backslash\{j\}) \overset{4.1.1}{=} \sigma_1([n])$$

for every $j \in [n]$. Since $Y_j = \sigma_1([n]\backslash\{j\})$ (see (4.3)), we can rewrite Y_j in the denominator of the fraction in equation (4.4). Hence for the complete left-hand side of the equation in the claim we get

$$\begin{aligned}
2q_2([n]) \cdot \prod_{j=1}^{n} (X_j + \sigma_1([n]\backslash\{j\})) &= \sum_{j=1}^{n} \frac{X_j \cdot Y_j}{\sigma_1([n])} \cdot \prod_{j=1}^{n} \sigma_1([n]) \\
&\overset{4.1.2}{=} 2\sigma_2([n])\sigma_1([n])^{n-1},
\end{aligned}$$

where we used Lemma 4.1.2 in the last step. □

Now we have a closer look at the transformations of the spanning tree polynomial if we exchange edges. Especially, the equation (4.4) plays an important role.

4.2.4 Construction. *Let $G = (V, E, \epsilon)$ be any (undirected) graph with a finite vertex-set V such that $|V| \geq 2$ and a finite edge-set E. We consider three types to exchange an edge in G.*

(A) Let $e \in E$ be an edge of the graph G with incident vertices r and s. We replace the edge e by $m \in \mathbb{N}$ parallel edges $e_1, \dots, e_m$ such that $e_1, \dots, e_m \notin E$.

Figure 4.1: Edge between the vertices r and s, on the left-hand side in Graph G and on the right-hand side replaced by m parallel edges in the new constructed graph G'.

We call the new graph with m parallel edges $G' = (V', E', \epsilon')$. This graph has almost the same vertex set V but we delete the edge e and add the new (pairwise disjoint) edges $e_1, \dots, e_m \notin E$. So the new edge-set is $E' = E \backslash \{e\} \dot\cup \{e_1, \dots, e_m\}$. The incident vertices of these edges are also r and s, i.e. $\epsilon'(e_i) = \{r, s\}$ for all $i \in [m]$.

Then the spanning tree polynomial $T_{G'}$ of the new graph G' is obtained by replacing X_e in the spanning tree polynomial T_G of the graph G by $\sum_{j=1}^{m} X_{e_j}$. This is because for every spanning tree T of G with $e \in E_T$, we get m different spanning trees $T_1, \dots, T_m$ of G' each using one of the edges $e_1, \dots, e_m$. The other edges of the spanning trees $T_1, \dots, T_m$ are the same as in T. All spanning trees T of G in which the edge e does not appear are also spanning trees of the new graph G'.

Another possibility to see how the spanning tree polynomial changes is to use the Laplacian. Let $L_G = (l_{ij})_{i,j \in [n]}$ be the Laplacian of G (3.2.1) and $L_{G'} = (l'_{ij})_{i,j \in [n]}$ the Laplacian of G'. The only entries changing are the two diagonal-entries corresponding to the vertices r and s and the two entries corresponding to the connection between the vertices r and s. These entries change exactly in the way that all edge-variables X_e in L_G are replaced by the sum $\sum_{j=1}^{m} X_{e_j}$ in $L_{G'}$.

(B) Instead of m parallel edges, we replace the edge $e \in E$ with $\epsilon(e) = \{r, s\}$ in G by a path r, e', v, e'', s.

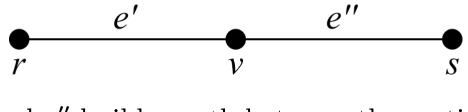


Figure 4.2: The edges e' and e'' build a path between the vertices r and s which replaces the edge e in the graph G.

The vertex v is supposed to be a new vertex and the edges e', e'' should not be contained in E. So $v \notin V$ and the new vertex-set considered for the new graph $\overline{G} = (\overline{V}, \overline{E}, \overline{\epsilon})$ is the set $\overline{V} = V \dot{\cup} \{v\}$. The edge-set of $\overline{G}$ is $\overline{E} = E \backslash \{e\} \dot{\cup} \{e', e''\}$ with $\overline{\epsilon}(e') = \{r, v\}$ and $\overline{\epsilon}(e'') = \{v, s\}$. The new spanning tree polynomial $T_{\overline{G}}$ of $\overline{G}$ is obtained by replacing the edge variable X_e of the initially edge e in T_G by $\frac{X_{e'}X_{e''}}{X_{e'}+X_{e''}}$ and multiply T_G with $X_{e'} + X_{e''}$. From the spanning trees of G, we obtain the spanning trees of the new constructed graph as follows. Each spanning tree of G containing the edge e gives a spanning tree of $\overline{G}$. For these spanning trees, the spanning tree polynomial changes by replacing X_e by $X_{e'}X_{e''}$. Considering the spanning trees of G without the edge e, we see that we must add a new edge, such that the new vertex v is connected to the others as well. There are two possibilities for this. Either we use e' or e'' to get a spanning tree of $\overline{G}$. So for each spanning tree of G without e we get two spanning trees of G'. This is the reason why we need to multiply the summands corresponding to those spanning trees of the spanning tree polynomial $T_{\overline{G}}$ with $X_{e'} + X_{e''}$. It is also possible to verify this by using the Laplacian of G and $\overline{G}$ and study how the entries differ. Please note that $\overline{G}$ has a vertex more than G such that the Laplacian $L_{\overline{G}}$ has one dimension more than L_G.

(C) *In the last step, we put (A) and (B) together and replace the edge $e \in E$ with incident vertices r and s by a series of $m \in \mathbb{N}$ parallel paths as in (B). We get a subgraph of the form*

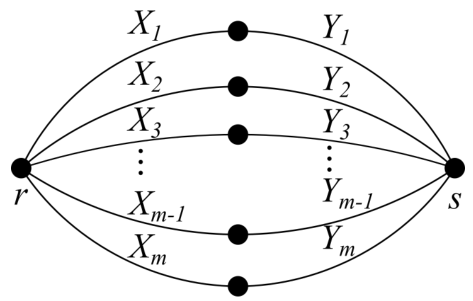


Figure 4.3: The edge e between the vertices r and s replaced by a series of m parallel paths. These new edges are assigned with the noted edge-variables.

between the vertices r and s with m new vertices and $2m$ new edges as in Figure 4.3. We first replace the edge e by $m \in \mathbb{N}$ parallel edges as in (A) and in the next step we replace each of these edges by a path of length 2 as in (B). The new spanning tree polynomial comes from the spanning tree polynomial T_G of G with X_e replaced by $\sum_{i=1}^{m} \frac{X_iY_i}{X_i+Y_i}$ and T_G multiplied by $\prod_{i=1}^{m}(X_i + Y_i)$.

It is not by chance that the transformation of the spanning tree polynomial in (C) looks like a part of equation (4.4). We use this similarity for the proof.

4.2.5 Proposition. *[Brä13, p.5]. The hyperbolicity cone of σ_2 is spectrahedral.*

Proof. We actually show that $\sigma_2\sigma_1^{m-1}$ is a determinantal polynomial. As seen in the construction

(C) the spanning tree polynomial of a graph as considered in Figure 4.3 is

$$\prod_{i=1}^{m}(X_i+Y_i)\sum_{i=1}^{m}\frac{X_i\cdot Y_i}{X_i+Y_i}.$$

This is by Proposition 4.2.3 exactly $\sigma_2\sigma_1^{n-1}$ with Y_j chosen as the linear transformation $Y_j = q_1([n]\backslash\{j\})$ for every $j\in[n]$. With the Matrix-Tree Theorem, we get that $\sigma_2\sigma_1^{n-1}$ is a determinantal polynomial because it is a spanning tree polynomial of a connected graph. Therefore it is spectrahedral (3.3.2). Note that a linear transformation of a spectrahedral cone stays a spectrahedral cone. To see that the hyperbolicity cone of σ_2 is spectrahedral, we need to show that $\overline{\Lambda}(\sigma_2,\mathbf{1})\subseteq\overline{\Lambda}(\sigma_1,\mathbf{1})$. This follows directly with the next lemma. So with Theorem 4.0.1 the proposition is shown. $\square$

4.2.6 Lemma. *For any $k\in[n-1]$ the hyperbolicity cones of the elementary symmetric polynomial σ_{k+1} is contained in the one of σ_k.*

Proof. We need to show $\Lambda(\sigma_{k+1},\mathbf{1})\subseteq\Lambda(\sigma_k,\mathbf{1})$. This is immediate by

$$\begin{aligned}
D_{\mathbf{1}}\sigma_{k+1} &= \sum_{i=1}^{n}\frac{\partial}{\partial X_i}\sum_{\substack{S\subseteq[n],\\ |S|=k+1}}\prod_{j\in S}X_j\\
&= \sum_{i=1}^{n}\sum_{\substack{S\subseteq[n],\\ |S|=k+1}}\underbrace{\frac{\partial}{\partial X_i}\prod_{j\in S}X_j}_{=0,\text{ if } i\notin S}\\
&= \sum_{i=1}^{n}\sum_{\substack{S\subseteq[n],\\ |S|=k+1,\\ i\in S}}\frac{\partial}{\partial X_i}\prod_{j\in S}X_j\\
&= \sum_{i=1}^{n}\sum_{\substack{S\subseteq[n],\\ |S|=k+1,\\ i\in S}}\prod_{j\in S\backslash\{i\}}X_j\\
&= \sum_{i=1}^{n}\underbrace{\sum_{\substack{S\subseteq[n]\backslash\{i\},\\ |S|=k}}\prod_{j\in S}X_j}_{=\sigma_k([n]\backslash\{i\})} = (n-k)\sigma_k.
\end{aligned}$$

This shows $\Lambda(\sigma_{k+1},\mathbf{1})\overset{1.3.5}{\subseteq}\Lambda(D_{\mathbf{1}}\sigma_{k+1},\mathbf{1})=\Lambda(\sigma_k,\mathbf{1})$ because σ_{k+1} hyperbolic in direction $\mathbf{1}$. $\square$

4.2.7 Remark. Although, we used a linear transformation of the variables, this does not change anything for a cone to be spectrahedral.

It is easy to see that the first elementary symmetric polynomial is spectrahedral. Now, we also showed that the second one is spectrahedral. The idea presented in the proof above, we want to use to show this for all elementary symmetric polynomials. For this, we need to make the construction of the graph in a higher degree. This we do by recursively exchange edges by graphs as in Figure 4.3.

4.3 Recursive construction of $G_{n,k}$

In this section, we want to define a graph such that the corresponding spanning tree polynomial fulfils the necessary conditions to apply Theorem 4.0.1. For this, we define recursively graphs $G_{n,k}$ for $n \geq k \geq 0$. A linear transformation of the corresponding spanning tree polynomial of $G_{n,k}$ has the $(k+1)$-th elementary-symmetric polynomial σ_{k+1} as a factor.

We have already seen the proof for the second elementary symmetric polynomial. Now, we want to extend this construction to higher degrees. For this, we recursively replace an edge in a graph by a subgraph as in Figure 4.3 and again by replacing one of those new edges by another subgraph. This we want to define more precisely in the following definition.

4.3.1 Definition. [Brä13, p.5]. We recursively construct a family of graphs $(G_{n,k})_{n\geq k\geq 0}$. For this recursion, we start with the base case $G_{n,0}$. This is a graph with two vertices and one connecting edge. So $G_{n,0} = (V_{n,0}, E_{n,0}, \epsilon_{n,0})$ has the vertex-set $V_{n,0} = \{s, z\}$ and edge-set $E_{n,0} = \{e\}$ such that $\epsilon_{n,0}(e) = \{s, z\}$.

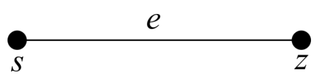


Figure 4.4: Graph $G_{n,0}$ consisting out of two vertices s and z and a connecting edge.

The graph $G_{n,k}$ for any $k \geq 1$ depends on the graph $G_{n,k-1}$. We obtain $G_{n,k}$ by replacing each edge $e \in E_{n,k-1}$ with $z \in \epsilon_{n,k-1}(e)$ by a graph as in Figure 4.3 with $m := n - k + 1$.

4.3.2 Example. Let's have a look, how these graphs look like. We consider the graph $G_{3,k}$ for $k = 1$ and $k = 2$. The graph $G_{3,0}$ is exactly the one drawn in Figure 4.4. Since this graph is independent of the integer n. In the second step, we replace the edge e of the graph $G_{3,0}$ by a graph as the one in Figure 4.3 with $m = 3 - 1 + 1 = 3$.

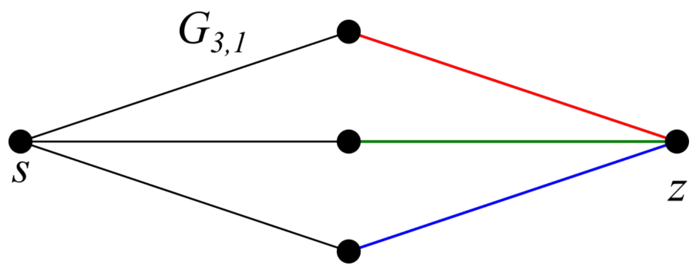


Figure 4.5: The graph $G_{3,1}$ with edges coloured which are incident to the vertex z.

The following graph $G_{3,2}$ is obtained by replacing each of the edges incident to z in $G_{3,1}$ (see the coloured edges in Figure 4.5) by another graph as in Figure 4.3 with $m = 3 - 2 + 1 = 2$

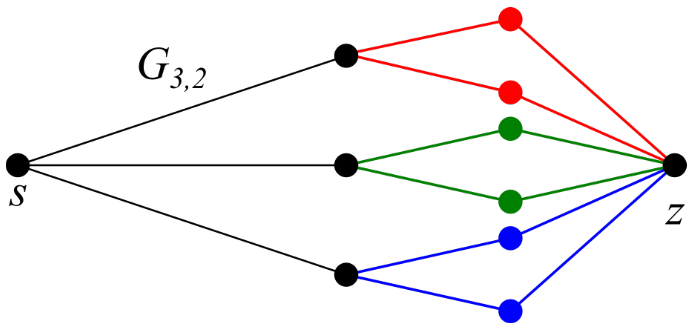


Figure 4.6: The graph $G_{3,2}$ constructed out of $G_{3,1}$ (see Figure 4.5).

Finally, the graph $G_{3,3}$ looks similar to the graph $G_{3,2}$ but the six edges incident to z are subdivided by another vertex.

This recursive construction is nice to see, but in order to work with these graphs it is also helpful to have a more explicit description of the graphs $G_{n,k}$ for $n \geq k \geq 0$. In the base case $k = 0$ we already achieved an explicit construction by definition.
So now, we are only interested in the case $n \geq k \geq 1$. For this reason, we label the vertices.

4.3.3 Proposition. *The graphs $G_{n,k}$ for $n \geq k \geq 1$ consist out of the vertices*

$$V_{n,k} = \{s, z\} \cup \{a : \ a = a_1 a_2 \cdots a_l \text{ word such that } 1 \leq l \leq k, a_j \in [n] \text{ for all } j \in [l] \\ \text{and } a_i \neq a_j \text{ for all } 1 \leq i < j \leq l\}$$

and edges are between the vertices

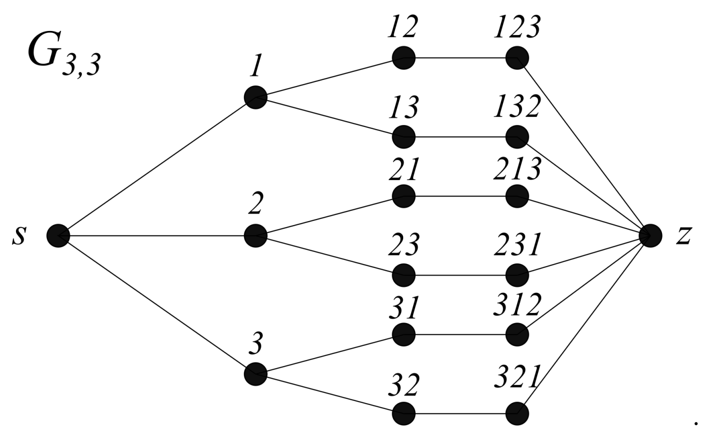
(1) s and i for all $1 \leq i \leq n$,

(2) $a_1 \cdots a_{i-1}$ and $a_1 \cdots a_{i-1} a_i$ for all $2 \leq i \leq k$ if $a_1 \cdots a_{i-1}, a_1 \cdots a_{i-1} a_i \in V_{n,k}$ and

(3) $a_1 \cdots a_k$ and z for all $a_1 \cdots a_k \in V_{n,k}$.

Proof. Follows by the construction of the graphs. □

4.3.4 Example. The graph $G_{3,3}$ with labelled vertices is:



For all $r \in \mathbb{N}_0$ with $k \leq r \leq n-1$, we consider edge-weights (n is fixed). In the base case, we define the weight of the edge e with $\epsilon(e) = \{s, z\}$ to be

$$(r+1)!q_{r+1}([n]) = (r+1)!\frac{\sigma_{r+1}}{\sigma_r}. \tag{4.5}$$

This is a rational function in $\mathbb{R}(\mathbf{X})$. The polynomial we are actually interested in is then a linear transformation of the standard spanning tree polynomial as we will see later on. The additional integer r, we need to get the recursion running. In the end, we are only interested in the spanning tree polynomial with $r = k$. First, we define a rational function which is a linear transformation of the spanning tree polynomial of the graphs $G_{n,k}$.

4.3.5 Definition. We define the rational function $H_{k,r}$ for any $0 \leq k \leq n$ and $r \geq k$ to be the spanning tree polynomial $T_{G_{n,k}}$ of the graph $G_{n,k}$ as defined above with edge-weight function $w: E_{n,k} \to \mathbb{R}(\mathbf{X})$, where

(a) $w(e) = r!X_i$ if $\epsilon(e) = (s, i)$ as in case (1) above,

(b) $w(e) = (r-k+1)!X_{a_i}$ if e is an edge between $a_1 \cdots a_{i-1}$ and $a_1 \cdots a_{i-1}a_i$ as in case (2) and

(c) $w(e) = (r-k+1)!q_{r-k+1}([n]\backslash\{a_1, \ldots, a_k\})$ for an edge e as in case (3) between the vertices $a_1 \cdots a_k$ and z or if $k = 0$ the edge-weight of the only existing edge (compare to 4.5).

4.3.6 Remark. We want to study the spanning tree polynomial of a graph using edge-weights instead of just assigned edge variables. These spanning tree polynomials come from the usual spanning tree polynomial as defined in 3.2.5 evaluated at $(w(e))_{e\in E}$. The result is not necessary a polynomial. In this case it is a rational function.

4.3.7 Remark. If we consider the rightmost piece of a graph $G_{n,k}$ with the edge-weights as mentioned in Definition 4.3.5 and we replace one of those subgraphs by a single edge, the new edge variable is the same as the one we would assign to the edge in $G_{n,k-1}$ using Definition 4.3.5 as we see soon.

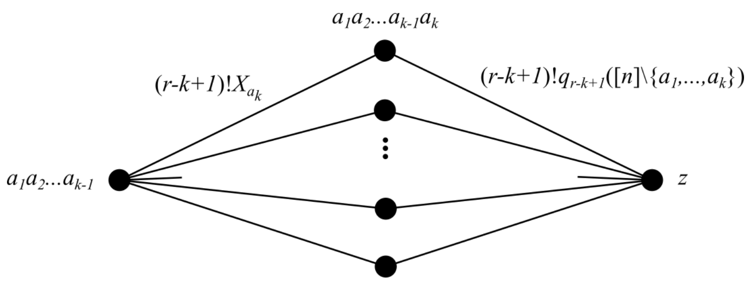


Figure 4.7: A rightmost piece of the graph $G_{n,k}$ with edge-weights.

With 4.2.4 (C), we get the edge-weight of $G_{n,k}$ with the above piece replaced as just a single edge between the vertices $a_1 a_2 \cdots a_{k-1}$ and z. It is

$$\begin{aligned}
&\sum_{a_k \in [n]\backslash\{a_1,\ldots,a_{k-1}\}} \frac{(r-k+1)! X_{a_k} \cdot (r-k+1)! q_{r-k+1}([n]\backslash\{a_1,\ldots,a_k\})}{(r-k+1)! X_{a_k} + (r-k+1)! q_{r-k+1}([n]\backslash\{a_1,\ldots,a_k\})} \\
= \quad & (r-k+1)! \sum_{a_k \in [n]\backslash\{a_1,\ldots,a_{k-1}\}} \frac{X_{a_k} \cdot q_{r-k+1}([n]\backslash\{a_1,\ldots,a_k\})}{X_{a_k} + q_{r-k+1}([n]\backslash\{a_1,\ldots,a_k\})} \\
\overset{4.2.2}{=} \quad & (r-k+1)! \cdot (r-k+2)\, q_{r-k+2}([n]\backslash\{a_1,\ldots a_{k-1}\}) \\
= \quad & (r-(k-1)+1)!\, q_{r-(k-1)+1}([n]\backslash\{a_1,\ldots a_{k-1}\}).
\end{aligned}$$

This is exactly the edge-weight we would assign to this edge in the graph $G_{n,k-1}$.

The spanning tree polynomial of the two graphs mentioned above are in general not equal. We need to multiply the polynomial with another factor (see 4.2.4 (C)).

4.4 Proof of the theorem

In this last section, we want to prove the theorem:

4.4.1 Theorem. *Hyperbolicity cones of elementary symmetric polynomials are spectrahedral.*

The proof presented for this theorem is orientated at the proof of Brändén [Brä13].

4.4.2 Definition. [Brä13, p.7]. To make the notation more readable, we introduce some shortcuts. We write for any finite set S and any integer $j \in \mathbb{N}$

$$\binom{S}{j} := \{U \subseteq S : \ |U| = j\}$$

for the set of all subsets of S of cardinality j. Furthermore for $0 \leq k \leq r \leq n$, we have

$$\gamma_{k,r} := \prod_{S \in \binom{[n]}{n-k}} \sigma_{r-k}(S)^{k!}.$$

4.4.3 Remark. It is easy to see that

$$\gamma_{k,r} \overset{\text{def.}}{=} \prod_{S \in \binom{[n]}{n-k}} \sigma_{r-k}(S)^{k!} = \prod_{S \in \binom{[n]}{k}} \sigma_{r-k}([n]\backslash S)^{k!}.$$

4.4.4 Lemma. *[Brä13, p.7, Lemma 2.3]. Let $0 \leq k \leq r \leq n-1$ be non-negative integers. Then there are positive constants $C_{k,r}$ such that*

$$H_{0,r} = C_{0,r} \frac{\sigma_{r+1}}{\sigma_r}$$

and

$$H_{k,r} = C_{k,r} H_{k-1,r} \frac{(\gamma_{k-1,r})^{n-k+1}}{\gamma_{k,r}}$$

for all $k > 0$.

Proof. The first statement follows directly from the definition of the graph $G_{n,k}$ and the edge-weight assigned in Definition 4.3.5. In this case, $C_{0,r}$ is the factorial $(r+1)!$ which is obviously positive.

For $k > 0$, we consider the graph $G_{n,k}$ with the specified edge-weights. As we have seen in 4.3.7, substitute a rightmost part of the graph by a single edge leads to the same edge-weight as the one in $G_{n,k-1}$. So if we substitute all of those rightmost parts, we get

$$H_{k,r} = Q_{k,r} \cdot H_{k-1,r},$$

where $Q_{k,r}$ is the product of all factors mentioned in 4.2.4 (C).
So now we want to determine the rational function $Q_{k,r}$.

For every word $a_1 a_2 \cdots a_k \in V_{n,k}$, we get the factor

$$\begin{aligned}
&(r-k+1)!X_{a_k} + (r-k+1)!q_{r-k+1}([n]\backslash\{a_1, a_2, \dots, a_k\}) \\
&= (r-k+1)!\left(X_{a_k} + \frac{\sigma_{r-k+1}([n]\backslash\{a_1, a_2, \dots, a_k\})}{\sigma_{r-k}([n]\backslash\{a_1, a_2, \dots, a_k\})}\right) \\
&= (r-k+1)!\frac{X_{a_k}\sigma_{r-k}([n]\backslash\{a_1, a_2, \dots, a_k\}) + \sigma_{r-k+1}([n]\backslash\{a_1, \dots, a_k\})}{\sigma_{r-k}([n]\backslash\{a_1, \dots, a_k\})} \\
&\overset{4.1.1}{=} (r-k+1)!\frac{\sigma_{r-k+1}([n]\backslash\{a_1, \dots, a_{k-1}\})}{\sigma_{r-k}([n]\backslash\{a_1, \dots, a_k\})}.
\end{aligned}$$

For k pairwise different elements $a_1, \dots, a_k \in [n]$ there are exactly $k!$ possibilities to order them. Hence the factor $\sigma_{r-k}([n]\backslash\{a_1, \dots, a_k\})$ appears exactly $k!$ times in the denominator of $Q_{k,r}$. This leads to the denominator $\gamma_{k,r}$ in the claim.

Just with the same argument, there are $(k-1)!$ possibilities to order the $a_1, \dots, a_{k-1}$ to a word, so the factor $\sigma_{r-k+1}([n]\backslash\{a_1, \dots, a_{k-1}\})$ appears $(k-1)!$ in the numerator of the rational function $Q_{k,r}$, which coincides with the factor $\gamma_{k-1,r}$ mentioned in the claim. For every word consisting of the elements $a_1, \dots, a_{k-1}$ there are $n-(k-1) = n-k+1$ possibilities to choose the next letter a_k to get a word of length k with pairwise different letters. This leads to the exponent $n-k+1$. The constant $C_{k,r}$ consists of the product of all factorials appearing, such that it is a positive integer. □

Furthermore, we are able to determine the polynomial $H_{k,k}$, which we are mainly interested in.

4.4.5 Lemma. *[Brä13, p.8, lemma 2.4]. For any integers $k \in \mathbb{N}$ with $1 \leq k \leq n-1$ the spanning tree polynomial $H_{k,k}$ has the elementary symmetric polynomial σ_{k+1} as a factor and we are also able to determine the other factors. We show that the factorisation is*

$$H_{k,k} = C_k \sigma_{k+1} \prod_{\substack{S \subseteq [n], \\ |S| \leq k-1}} \left(\partial^S \sigma_k\right)^{|S|!(n-|S|-1)},$$

where C_k is a positive constant and $\partial^S \sigma_k = \left(\prod_{j \in S} \frac{\partial}{\partial X_j}\right) \sigma_k$ as defined in 4.1.3.

Proof. By iterated application of Lemma 4.4.4, we see that the polynomial $H_{r,r}$ for any $r \leq n-1$ looks like

$$H_{r,r} = c_r \left(\prod_{j=1}^{r} \frac{(\gamma_{r-j,r})^{n-r+j}}{\gamma_{r-j+1,r}} \right) H_{0,r}, \tag{4.6}$$

where c_r is the product of all positive constants $C_{j,r}$ for $j \in \{1, \ldots, r\}$, so it is again a positive constant. The product in equation (4.6) can be simplified as

$$\begin{aligned} &\prod_{j=1}^{r} \frac{(\gamma_{r-j,r})^{n-r+j}}{\gamma_{r-j+1,r}} \\ &= \frac{(\gamma_{r-1,r})^{n-r+1}}{\gamma_{r,r}} \cdot \frac{(\gamma_{r-2,r})^{n-r+2}}{\gamma_{(r-1),r}} \cdots \frac{(\gamma_{1,r})^{n-2}}{\gamma_{2,r}} \cdot \frac{(\gamma_{0,r})^{n}}{\gamma_{1,r}} \\ &= \frac{(\gamma_{0,r})^{n}}{\gamma_{r,r}} \prod_{j=1}^{r-1} \left(\gamma_{(r-j),r}\right)^{n-r+j-1}. \end{aligned} \tag{4.7}$$

With the definition of $\gamma_{k,r}$ (Definition 4.4.2), we get

$$\gamma_{0,r} = \prod_{S \in \binom{[n]}{n}} \sigma_r(S) = \sigma_r \text{ and} \tag{4.8}$$

$$\gamma_{r,r} = \prod_{S \in \binom{[n]}{n-r}} \underbrace{\sigma_0(S)}_{=1}{}^{r!} = 1. \tag{4.9}$$

Another application of Lemma 4.4.4 for $H_{0,r}$ gives us

$$H_{0,r} = C_{0,r} \frac{\sigma_{r+1}}{\sigma_r}. \tag{4.10}$$

Inserting (4.7) - (4.10) in (4.6) shows

$$H_{r,r} = C_r \cdot \sigma_{r+1} \prod_{j=1}^{r} (\gamma_{r-j+1,r})^{n-r+j-1} = C_r \cdot \sigma_{r+1} \prod_{j=0}^{r-1} (\gamma_{j,r})^{n-j-1}. \tag{4.11}$$

As we think of Lemma 4.1.5, we are able to rewrite the part $\partial^S \sigma_k$ as

$$\partial^S \sigma_r = \sigma_{r-|S|}([n] \backslash S) \tag{4.12}$$

for all $S \subseteq [n]$ such that $|S| \leq r$. So all in all we get

$$\begin{aligned} \prod_{j=0}^{r-1} (\gamma_{j,r})^{n-j-1} \overset{4.4.3}{=} \prod_{j=0}^{r-1} \left(\prod_{S \in \binom{[n]}{j}} \sigma_{r-j}([n] \backslash S)^{j!} \right)^{n-j-1} &= \prod_{\substack{S \subseteq [n] \\ |S| \leq r-1}} \sigma_{r-|S|}([n] \backslash S)^{|S|!(n-|S|-1)} \\ &\overset{(4.12)}{=} \prod_{\substack{S \subseteq [n] \\ |S| \leq r-1}} \partial^S \sigma_r([n] \backslash S)^{|S|!(n-|S|-1)}. \end{aligned}$$

So after all the transformations, we finally get the claim. $\square$

4.4.6 Remark. We are only interested in the spanning tree polynomial $H_{k,k}$ but still we introduced a new integer r and defined the spanning tree polynomial $H_{k,r}$. The reason that we were in need of this integer r for the edge-weights was the last lemma. In the proof for this lemma we used recursively the previous lemma $H_{j,r}$ with decreasing j.

4.4.7 Definition. For any open and connected set $U \subseteq \mathbb{R}^n$, $\mathscr{P}_{n,m}(U)$ denotes the space of all hyperbolic polynomials of degree $m \in \mathbb{N}$ in $\mathbb{R}[\mathbf{X}] = \mathbb{R}[X_1, \ldots, X_n]$ such that $p \in \mathscr{P}_{n,m}(U)$ if and only if for all $\mathbf{d} \in U$ the polynomial p is hyperbolic in direction $\mathbf{d}$.

4.4.8 Remark. In the definiton above, we assumed U is connected. This is the reason why for all $p \in \mathscr{P}_{n,m}(U)$ there is a direction $\mathbf{d}$ such that $U \subseteq \Lambda(p, \mathbf{d})$.

4.4.9 Lemma. *[Brä13, Lemma 2.5]. For any open and connected set $U \subseteq \mathbb{R}^n$ the space $\mathscr{P}_{n,m}(U) \cup \{0\}$ is closed under point-wise convergence.*

For the proof of this lemma, we need two theorems from complex analysis. For the proof of Rouchés Theorem, we refer to [APP11, Theorem 37.2].

4.4.10 Lemma (Rouché's Theorem)**.** *Let f and g be two analytic functions on a domain, i.e. a connected open subset $D \subseteq \mathbb{C}$ such that*

$$|f(z)| > |g(z)|$$

for all z on a simple closed contour γ in D. Then f and $f + g$ have the same number of zeros inside the contour γ (with multiplicities).

4.4.11 Lemma (Hurwitz's Theorem)**.** *[Cho+02, p.96] and [APP11, Theorem 37.4]. If D is a domain, i.e. a connected open subset of $\mathbb{C}^n$ and $(f_k)_{k\in\mathbb{N}}$ a sequence of non-vanishing analytic functions on D converging uniformly to f on all compact subsets of D, then f is either non-vanishing on D or $f = 0$.*

Proof. First, we show the special case for $n = 1$ of Hurwitz's Theorem. We need to prove that $f(z) \neq 0$ for all $z \in D \subseteq \mathbb{C}$ if $f \neq 0$. Assume $f \neq 0$ but $f(a) = 0$ for an $a \in D$. We want to show that there is an $n_0 \in \mathbb{N}$ such that f_n has a zero in D for all $n \geq n_0$.

Let $\delta > 0$ such that $B_\delta(a) := \{x \in \mathbb{C} : |x - a| < \delta\} \subseteq D$ and $f\,\big|_{\partial B_\delta(a)}$ never vanishes. Now, take $\varepsilon := \inf\limits_{z\in\partial B_\delta(a)} |f(z)| > 0$. Since $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $z \in \partial B_\delta(a)$

$$|f_n(z) - f(z)| < \varepsilon.$$

Furthermore, by definition of ε the inequality $\varepsilon \leq |f(z)|$ for all $z \in \partial B_\delta(a)$ holds. Both together implies

$$|f_n(z) - f(z)| < |f(z)|$$

for all $z \in \partial B_\delta(a)$ and all $n \geq n_0$. Rouché's Theorem 4.4.10 implies that f and $(f_n - f) + f = f_n$ have the same number of zeros in $\partial B_\delta(a)$ for all $n \geq n_0$. So f_n has a root in D. This is a contradiction to the assumption. So the special case for $n = 1$ is proved.

Now, let $n \in \mathbb{N}$ be an arbitrary integer. Again, we assume there is a zero $a = (a_1, \ldots, a_n) \in$

$D \subseteq \mathbb{C}^n$ with $f(a) = 0$. Again, we find a ball $B_\delta(a) \subseteq D \subseteq \mathbb{C}^n$. Applying Hurwitz's Theorem for $n = 1$, we see that $f(z_1, a_2, \dots, a_n) = 0$ for all $z_1 \in \mathbb{C}$ such that $(z_1, a_2, \dots, a_n) \in B_\delta(a)$. For all those n-tuples $(z_1, a_2, \dots, a_n)$, we apply Hurwitz's Theorem for the case $n = 1$ again and get $f(z_1, z_2, a_3, \dots, a_n) = 0$ for all $z_2 \in \mathbb{C}$ such that $(z_1, z_2, a_3, \dots, a_n) \in B_\delta(a)$. Repeated application shows $f\big|_{B_\delta(a)} = 0$. Since f is analytic $f = 0$ in D. □

Proof of Lemma 4.4.9. First, we show that a homogeneous polynomial $p \in \mathbb{R}[\mathbf{X}]$ of degree m belongs to $\mathscr{P}_{n,m}(U)$ if and only if for all $\mathbf{z} \in U + i\mathbb{R}^n := \{\mathbf{x} + i\mathbf{y} : \mathbf{x} \in U \wedge \mathbf{y} \in \mathbb{R}^n\}$ the polynomial is non-vanishing, i.e. $p(\mathbf{z}) \neq 0$.

If $p \in \mathscr{P}_{n,m}(U)$, the polynomial p is hyperbolic in direction $\mathbf{d}$ for every $\mathbf{d} \in U$. Then for every $\mathbf{z} = \mathbf{d} + i\mathbf{y} \in U + i\mathbb{R}^n$, we get $p(\mathbf{z}) = p(\mathbf{d} + i\mathbf{y}) = (-i)^m p(-\mathbf{y} + i\mathbf{d}) \neq 0$ by the homogeneity and the definition of the hyperbolicity because for a hyperbolic polynomial p in direction $\mathbf{d}$ the univariate polynomial $p(\mathbf{x} + T\mathbf{d})$ has only real zeros if $\mathbf{x}$ is a real vector.

Conversely, if p is not in $\mathscr{P}_{n,m}(U)$, there is a direction $\mathbf{d} \in U$ such that p is not hyperbolic in direction $\mathbf{d}$. So $p(\mathbf{x} + T\mathbf{d})$ has a zero $a + ib \in \mathbb{C}$ with $a, b \in \mathbb{R}$ and $b \neq 0$ for any $\mathbf{x} \in \mathbb{R}^n$. So

$$0 = p(\mathbf{x} + (a + ib)\mathbf{d}) = (ib)^m p(\mathbf{d} - ib^{-1}\mathbf{x} - aib^{-1}\mathbf{d})$$

such that $p(\mathbf{d} - i(\underbrace{b^{-1}\mathbf{x} + ab^{-1}\mathbf{d}}_{\in\mathbb{R}^n})) = 0$, so p fails to be non-vanishing on $U + i\mathbb{R}^n$.

Now let $(p_k)_{k\in\mathbb{N}}$ be a sequence of polynomials in $\mathscr{P}_{n,m}(U)$ which converges point-wise to p, i.e

$$\forall \mathbf{x} \in \mathbb{C}^n : \lim_{k\to\infty} |p_k(\mathbf{x}) - p(\mathbf{x})| = 0. \tag{4.13}$$

The space of all homogeneous polynomials of degree m unified with the zero polynomial is closed under point-wise convergence, so p is a homogeneous polynomial of degree m if not the zero polynomial and it remains to prove that p is hyperbolic in every direction $\mathbf{e} \in U$ or $p = 0$ (in $\mathbb{R}[\mathbf{X}]$).

To show this, we want to apply Hurwitz' Theorem 4.4.11. We choose $D = U + i\mathbb{R}^n$ to be the domain. For all $k \in \mathbb{N}$ the polynomials p_k are in $\mathscr{P}_{n,m}(U)$, so by the statement shown above all p_k are non-vanishing on D. This implies that for all $\mathbf{z} \in D$, $\|\cdot\|_{\mathbf{z}} : \mathscr{P}_{n,m}(U) \cup \{0\} \to \mathbb{R}_{\geq 0}, p \mapsto \|p\|_{\mathbf{z}} = |p(\mathbf{z})|$ is a norm on $\mathscr{P}_{n,m}(U) \cup \{0\}$. Furthermore, for every compact subset $C \subseteq D$, $\|\cdot\|_C : \mathscr{P}_{n,m}(U) \to \mathbb{R}_{\geq 0} \geq 0, p \mapsto \|p\|_C = \sup_{\mathbf{z}\in C} |p(\mathbf{z})| = \max_{\mathbf{z}\in C} |p(\mathbf{z})|$ is another norm on $\mathscr{P}_{n,m}(U)$. On a finite-dimensional vector space all norms are equivalent, so with $p_k \overset{k\to\infty}{\to} p$ point-wise, p_k converges also uniformly to p on any compact subset $C \subseteq D$. Now, we are able to apply the Hurwitz' Theorem. This says that p is either non-vanishing on D or $p = 0$ in D. The first part says by the statement above $p \in \mathscr{P}_{n,m}(U)$. Hence in each case, we get $p \in \mathscr{P}_{n,m}(U) \cup \{0\}$. It is shown that the limit of an arbitrary point-wise converging sequence is in the set itself, so it is closed under point-wise convergence. □

4.4.12 Lemma. *[Brä13, Lemma 2.6]. Let $p \in \mathbb{R}[\mathbf{X}]$ be a hyperbolic polynomial with respect to the direction $\mathbf{d} \in \mathbb{R}^n$ and let v be any point from the hyperbolicity cone $\overline{\Lambda}(p, \mathbf{d})$ such that $D_{\mathbf{v}}p \neq 0$ (in $\mathbb{R}[\mathbf{X}]$). Then the directional derivative $D_{\mathbf{v}}p$ is hyperbolic in direction $\mathbf{d}$ and $\overline{\Lambda}(p, \mathbf{d}) \subseteq \overline{\Lambda}(D_{\mathbf{v}}p, \mathbf{d})$.*

Proof. Consider a polynomial p, hyperbolic with respect to $\mathbf{d} \in \mathbb{R}^n$ and any point $v \in \overline{\Lambda}(p, \mathbf{d})$ such that $D_{\mathbf{v}}p \neq 0$. We split the proof into two parts.

In the first case, we assume that v is in an interior point of the (closed) hyperbolicity cone $\overline{\Lambda}(p, \mathbf{d})$. By Theorem 1.2.14, it follows that p is hyperbolic in direction v and $\Lambda(p, \mathbf{d}) = \Lambda(p, \mathbf{v})$. Now, we are able to apply Proposition 1.3.5 and get the inclusion $\Lambda(p, \mathbf{v}) \subseteq \Lambda(D_{\mathbf{v}}p, \mathbf{v})$. Since $\mathbf{d} \in \Lambda(p, \mathbf{d}) = \Lambda(p, \mathbf{v})$, the direction $\mathbf{d}$ is also in the open hyperbolicity cone $\Lambda(D_{\mathbf{v}}p, \mathbf{v})$ such that Theorem 1.2.14 says that $D_{\mathbf{v}}p$ is hyperbolic in direction $\mathbf{d}$ and $\Lambda(D_{\mathbf{v}}p, \mathbf{v}) = \Lambda(D_{\mathbf{v}}p, \mathbf{d})$. Altogether, it follows $\Lambda(p, \mathbf{d}) \subseteq \Lambda(D_{\mathbf{v}}p, \mathbf{d})$ and this implies the inclusion for the closures $\overline{\Lambda}(p, \mathbf{d}) \subseteq \overline{\Lambda}(D_{\mathbf{v}}p, \mathbf{d})$.

In the second case, we consider $\mathbf{v}$ to be a point of the boundary of $\overline{\Lambda}(p, \mathbf{d})$. This means by definition of the boundary, that there is a sequence $(\mathbf{v}_k)_{k\in\mathbb{N}} \subseteq \Lambda(p, \mathbf{d})$ such that $\mathbf{v}_k \overset{k\to\infty}{\to} \mathbf{v}$. As shown above, $D_{\mathbf{v}_k}p \in \mathscr{P}_{n,m}(\Lambda(p, \mathbf{d}))$ for all $k \in \mathbb{N}$ and obviously $D_{\mathbf{v}_k}p \overset{k\to\infty}{\to} D_{\mathbf{v}}p$ (point-wise). By Lemma 4.4.9 the space $\mathscr{P}_{n,m}(\Lambda(p, \mathbf{d})) \cup \{0\}$ is closed under point-wise convergence, such that $D_{\mathbf{v}}p \in \mathscr{P}_{n,m}(\Lambda(p, \mathbf{d})) \cup \{0\}$. We assumed that $D_{\mathbf{v}}p \neq 0$, so $D_{\mathbf{v}}p \in \mathscr{P}_{n,m}(\Lambda(p, \mathbf{d}))$. This means for any $\mathbf{x} \in \Lambda(p, \mathbf{d})$, the polynomial $D_{\mathbf{v}}p$ is hyperbolic in direction $\mathbf{d}$ and so $\mathbf{x} \in \Lambda(D_{\mathbf{v}}p, \mathbf{d})$. □

With all the previous work, we are finally able to prove the main result of this work.

4.4.13 Theorem. *The hyperbolicity cones of elementary-symmetric polynomials are spectrahedral.*

Proof of the theorem. It is easy to show that σ_1 is spectrahedral because this is a linear polynomial (see 1.2.9 (1)). So WLOG, we choose an elementary symmetric polynomial σ_{k+1} for any $k \in [n-1]$. Now consider the graph $G_{n,k} = (V, E, \epsilon)$ as defined in 4.3.1. The corresponding spanning tree polynomial is

$$H_{k,k} = C_k \sigma_{k+1} \prod_{\substack{S\subseteq[n],\\ |S|\leq k-1}} \left(\partial^S \sigma_k\right)^{|S|!(n-|S|-1)},$$

see 4.4.5. With the Matrix-Tree Theorem 3.2.6, we know there are positive semi-definite matrices $(A_e)_{e\in E}$ such that $H_{k,k} = \det\left(\sum_{e\in E} X_e A_e\right)$. If we evaluate the spanning tree polynomial $H_{k,k}$ at $\mathbf{1} = (1)_{e\in E}$, we get the number of spanning tree of $G_{n,k}$. Since $G_{n,k}$ is connected, there is at least one spanning tree. Hence

$$\det\left(\sum_{e\in E} A_e\right) \in \mathbb{N}.$$

So $A := \sum_{e\in E} A_e$ is a positive definite matrix.

A more detailed look at the definition of the edge-weights assigned to the edges in the graph $G_{n,k}$ (Definition 4.3.5) for $r = k$ shows that the weight of the edge incident to z and $a_1 \cdots a_k$ is

$$q_1([n]\backslash\{a_1, \dots, a_k\}) = \sigma_1([n]\backslash\{a_1, \dots, a_k\}) = \sum_{j\in[n]\backslash\{a_1,\dots,a_k\}} X_j.$$

They are only linear in the variables $X_1, \dots, X_n$, so there is a possibility to write $H_{k,k} = \det \sum_{j=1}^{n} X_j B_j$ with positive semi-definite matrices $B_1, \dots, B_n$. The matrix $B := \sum_{j=1}^{n} B_j$ is positive definite because $\det B = \det\left(\sum_{j=1}^{n} B_j\right) = H_{k,k}(\mathbf{1}) \neq 0$. It remains to prove the inclusion $\Lambda(\sigma_{k+1}, \mathbf{1}) \subseteq \Lambda(\partial^S \sigma_k, \mathbf{1})$ for any subset $S \subseteq [n]$ with $|S| \leq k-1$. Then we apply Theorem 4.0.1 and conclude the claim. This is enough because the hyperbolicity cone of a product of hyperbolic polynomials is the intersections of the hyperbolicity cone of its factors 1.3.1.

The subset relation $\Lambda(\sigma_{k+1}, \mathbf{1}) \subseteq \Lambda(\sigma_k, \mathbf{1})$, we have already shown in Lemma 4.2.6. Since the open hyperbolicity cone $\Lambda(\sigma_k, \mathbf{1})$ of any elementary symmetric polynomial contains the whole orthant including the vector $\mathbf{1}$, the closure $\overline{\Lambda}(\sigma_k, \mathbf{1})$ contains the positive coordinate axis and especially $\mathbf{e}_k$ (standard basis vector) for every $k \in [n]$. Furthermore by definition, it is $\partial^S \sigma_k = \left(\prod_{i \in S} D_{\mathbf{e}_i}\right) \sigma_k$, so with repeated application of Lemma 4.4.12 it holds $\overline{\Lambda}(\sigma_k, \mathbf{1}) \subseteq \overline{\Lambda}(\partial^S \sigma_k, \mathbf{1})$. □

This theorem was the aim of this thesis but still we are going to state an important corollary. But first we need to prove another lemma.

4.4.14 Lemma. *[Ren06, Proposition 18]. Let $p \in \mathbb{R}[\mathbf{X}]$ be a hyperbolic polynomial of degree $m \in \mathbb{N}$, hyperbolic with respect to $\mathbf{d} \in \mathbb{R}^n$. For each $0 \leq k \leq m$ the k-th directional derivative of p in direction $\mathbf{d} \in \mathbb{R}^n$ is determined by*

$$D_{\mathbf{d}}^{(k)} p(\mathbf{x}) = k! p(\mathbf{d}) \sigma_{m-k}(\lambda_1(\mathbf{d}, \mathbf{x}), \ldots, \lambda_m(\mathbf{d}, \mathbf{x}))$$

for every $\mathbf{x} \in \mathbb{R}^n$. In this case the elementary symmetric polynomials σ_{m-k}, $0 \leq k \leq m$ are in m variables.

Proof. As we have seen in the proof of 1.3.5 for any polynomial p the evaluation of the directional derivative $(D_{\mathbf{d}} p)(\mathbf{x} + T\mathbf{d})$ is the same as the usual formal derivative of $p(\mathbf{x} + T\mathbf{d})$. By induction, it follows

$$(D_{\mathbf{d}}^{(k)} p)(\mathbf{x} + T\mathbf{d}) = (p(\mathbf{x} + T\mathbf{d}))^{(k)}$$

for any integer k with $0 \leq k \leq m$. So if we recall the presentation of $p(\mathbf{x} + T\mathbf{d})$, we get

$$p(\mathbf{x} + T\mathbf{d}) = p(\mathbf{d}) \prod_{j=1}^{m} (T + \lambda_j(\mathbf{d}, \mathbf{x})) = p(\mathbf{d}) \sum_{j=0}^{m} T^j \sigma_{m-j}(\lambda_1(\mathbf{d}, \mathbf{x}), \ldots, \lambda_m(\mathbf{d}, \mathbf{x})).$$

In this case the elementary-symmetric polynomials are in m variables So for the k-th directional derivative it follows

$$\begin{aligned}(D_{\mathbf{d}}^{(k)} p)(\mathbf{x} + T\mathbf{d}) = (p(\mathbf{x} + T\mathbf{d}))^{(k)} &= p(\mathbf{d}) \sum_{j=k}^{m} \frac{j!}{(j-k)!} T^{j-k} \sigma_{m-j}(\lambda_1(\mathbf{d}, \mathbf{x}), \ldots, \lambda_m(\mathbf{d}, \mathbf{x})) \\ &= p(\mathbf{d}) \sum_{j=0}^{m-k} \frac{(j+k)!}{j!} T^j \sigma_{m-j-k}(\lambda_1(\mathbf{d}, \mathbf{x}), \ldots, \lambda_m(\mathbf{d}, \mathbf{x})). \qquad (4.14)\end{aligned}$$

Evaluating the univariate polynomial in equation (4.14) at 0 shows

$$\left((D_{\mathbf{d}}^{(k)} p)(\mathbf{x} + T\mathbf{d})\right)(0) = p(\mathbf{d}) k! \sigma_{m-k}(\lambda_1(\mathbf{d}, \mathbf{x}), \ldots, \lambda_m(\mathbf{d}, \mathbf{x})).$$

□

As we have seen in Example 1.2.9 (1), the hyperbolicity cones of polynomials of the form $p = \prod_{j=1}^{m} l_j$ for linear polynomials $l_j \in \mathbb{R}[\mathbf{X}]$ are polyhedral. So it is natural to ask what happens to the derivative cone of a polyhedral cone. The first derivative cone was studied by Sanyal [San11]. Using the Theorem 4.4.13, we are finally able to answer this question for any $k \in [m-1]$.

4.4.15 Corollary. *The derivative cones of polyhedrals are spectrahedral. This means, for a homogeneous polynomial $p \in \mathbb{R}[X_1, \dots, X_n]$ with linear factorisation $p = \prod_{j=1}^{m} l_j$, where all $l_j \in \mathbb{R}[X_1, \dots, X_n]$, $j \in [m]$, are linear and homogeneous with $p(\mathbf{d}) \neq 0$ for a point $\mathbf{d} \in \mathbb{R}^n$, the derivative cone $\Lambda(D_{\mathbf{d}}^{(k)} p, \mathbf{d})$ is spectrahedral for all $1 \leq k \leq m-1$.*

Proof. The polynomial p is the m-th elementary symmetric polynomial in m variables evaluated in the m linear homogeneous polynomials $l_1, \dots, l_m$. This means

$$p = \sigma_m(l_1, \dots, l_m).$$

Since $p(\mathbf{d}) \neq 0$, no linear factor l_i, $i \in [m]$ vanishes at the point $\mathbf{d} \in \mathbb{R}^n$, so $\sigma_m(l_1(\mathbf{d}), \dots, l_m(\mathbf{d}) \neq 0$, which implies that σ_m is hyperbolic in direction $(l_1(\mathbf{d}), \dots, l_m(\mathbf{d})$. From this we can conclude with

$$\begin{aligned} p(\mathbf{x} + T\mathbf{d}) &= \sigma_m(l_1(\mathbf{x} + T\mathbf{d}), \dots, l_m(\mathbf{x} + T\mathbf{d})) \\ &= \sigma_m(l_1(\mathbf{x}) + Tl_1(\mathbf{d}), \dots, l_m(\mathbf{x}) + Tl_m(\mathbf{d})) \\ &= \sigma_m(l_1(\mathbf{d}), \dots, l_m(\mathbf{d})) \prod_{j=1}^{m} (T + \lambda_j\left((l_1(\mathbf{d}), \dots, l_m(\mathbf{d})), (l_1(\mathbf{x}), \dots, l_m(\mathbf{x}))\right) \end{aligned}$$

that p is hyperbolic in direction $\mathbf{d}$. Since σ_m is hyperbolic, the eigenvalues $\lambda_j\left((l_1(\mathbf{d}), \dots, l_m(\mathbf{d})), \mathbf{y}\right)$ are real for all $\mathbf{y} \in \mathbb{R}^m$ and because the linear polynomials l_i have real coefficients the point $(l_1(\mathbf{x}), \dots, l_m(\mathbf{x}))$ is real for every $\mathbf{x} \in \mathbb{R}^n$. Therefore the eigenvalues of p are real.

Applying L Lemma 4.4.14, we get for the k-th directional derivative in direction $\mathbf{d}$ the presentation:

$$D_{\mathbf{d}}^{(k)} p = k! p(\mathbf{d}) \sigma_{m-k}(l_1, \dots, l_m).$$

for any $k \in [m-1]$.
By theorem the hyperbolicity cone $\Lambda(\sigma_m, (l_1(\mathbf{d}), \dots, l_m(\mathbf{d})))$ is spectrahedral. So there are symmetric matrices $A_1, \dots A_m$ such that there exists a vector $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$ with $\sum_{i=1}^{m} y_i A_i \succ 0$ and $\overline{\Lambda}(\sigma_m, (l_1(\mathbf{d}), \dots, l_m(\mathbf{d}))) = \{\mathbf{x} \in \mathbb{R}^m : \sum_{i=1}^{m} x_i A_i \succeq 0\}$. Using this representation

of the hyperbolicity cone, we see that

$$\begin{aligned}\overline{\Lambda}(\mathbf{d}, \sigma_m(l_1, \ldots, l_m)) &= \{\mathbf{x} \in \mathbb{R}^n : \, l(\mathbf{x}) \in \overline{\Lambda}((l_1(\mathbf{d}), \ldots, l_m(\mathbf{d})), \sigma_m)\} \\ &= \left\{\mathbf{x} \in \mathbb{R}^n : \, \sum_{i=1}^{m} l_i(\mathbf{x}) A_i \succeq 0\right\} \\ &= \left\{\mathbf{x} \in \mathbb{R}^n : \, \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} c_{i,j} A_i\right) \succeq 0\right\} \\ &= \left\{\mathbf{x} \in \mathbb{R}^n : \, \sum_{j=1}^{n} x_j B_j \succeq 0\right\}\end{aligned}$$

where we used $l_i = \sum_{j=1}^{n} c_{i,j} X_j$ for all $i \in [m]$ and some real coefficients $c_{i,j}$. Furthermore $B_j := \sum_{i=1}^{m} c_{i,j} A_i$ are symmetric matrices for all $j \in [n]$.

It remains to show that there is a vector $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ such that $\sum_{j=1}^{n} z_j B_j \succ 0$ which follows directly from the fact that

$$(l_1(\mathbf{d}), \ldots, l_m(\mathbf{d})) \in \Lambda(\sigma_k, (l_1(\mathbf{d}), \ldots, l_m(\mathbf{d}))) = \left\{\mathbf{x} \in \mathbb{R}^m : \, \sum_{i=1}^{m} x_i A_i \succ 0\right\}$$

see 1.2.3. $\square$

Bibliography

[APP11] Ravi P. Agarwal, Kanishka Perera, and Sandra Pinelas. *An Introduction to Complex Analysis*. Springer, 2011. ISBN: 978-1-4614-0194-0.

[Bos09] Siegfried Bosch. *Algebra*. Springer Berlin Heidelberg, Apr. 3, 2009. ISBN: 978-3-540-92812-6. DOI: 10.1007/978-3-642-39567-3.

[Brä13] Petter Brändén. "Hyperbolicity cones of elementary symmetric polynomials are spectrahedral". In: (Sept. 20, 2013). DOI: 10.1007/s11590-013-0694-6. arXiv: 1204.2997v2 [math.OC].

[Bro13] M. Brodmann. *Algebraische Geometrie: Eine Einführung*. Basler Lehrbücher. Birkhäuser Basel, 2013. ISBN: 9783034892667.

[Cho+02] Young-Bin Choe et al. "Homogeneous multivariate polynomials with the half-plane property". In: *Adv. Appl. Math. 32, 88-187 (2004)* (Feb. 5, 2002). DOI: 10.1016/S0196-8858(03)00078-2. arXiv: math/0202034v2 [math.CO].

[CK78] S Chaiken and D.J Kleitman. "Matrix Tree Theorems". In: *Journal of Combinatorial Theory, Series A* 24.3 (May 1978), pp. 377–381. DOI: 10.1016/0097-3165(78)90067-5.

[DR11] Robert Denk and Reinhard Racke. *Kompendium der ANALYSIS - Ein kompletter Bachelor-Kurs von Reellen Zahlen zu Partiellen Differentialgleichungen*. Vieweg+Teubner Verlag, May 16, 2011.

[Går59] Lars Gårding. "An inequality for hyperbolic polynomials". In: *J. Math. Mech.* 8 (1959), pp. 957–965. DOI: 10.1512/iumj.1959.8.58061.

[Gül97] Osman Güler. "Hyperbolic polynomials and interior point methods for convex programming." English. In: *Math. Oper. Res.* 22.2 (1997), pp. 350–377. ISSN: 0364-765X; 1526-5471/e. DOI: 10.1287/moor.22.2.350.

[GY98] Jonathan L. Gross and Jay Yellen. *Graph Theory and Its Applications*. CRC Press, 1998. ISBN: 0849339820. DOI: 10.2307/3621555.

[HV07] J. William Helton and Victor Vinnikov. "Linear matrix inequality representation of sets". In: *Communications on Pure and Applied Mathematics* 60.5 (2007), pp. 654–674. ISSN: 1097-0312. DOI: 10.1002/cpa.20155.

[KPV15] Mario Kummer, Daniel Plaumann, and Cynthia Vinzant. "Hyperbolic polynomials, interlacers, and sums of squares". In: *Mathematical Programming* 153.1 (Oct. 2015), pp. 223–245. ISSN: 1436-4646. DOI: 10.1007/s10107-013-0736-y.

[LPR03] Adrian S. Lewis, Pablo A. Parrilo, and Motakuri V. Ramana. "The Lax conjecture is true". In: *Proceedings of the AMS, Vol. 133, pp. 2495-2499, 2005.* (Apr. 8, 2003). DOI: 10.1090/S0002-9939-05-07752-X. arXiv: math/0304104v2 [math.OC].

[NS12] Tim Netzer and Raman Sanyal. “Smooth Hyperbolicity Cones are Spectrahedral Shadows”. In: (Aug. 2, 2012). DOI: 10.1007/s10107-014-0744-6. arXiv: 1208.0441v1 [math.OC].

[Pri13] S. Priess-Crampe. *Angeordnete Strukturen: Gruppen, Körper, projektive Ebenen.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge. Springer Berlin Heidelberg, 2013. ISBN: 9783642686283.

[Ren06] James Renegar. “Hyperbolic programs, and their derivative relaxations.” English. In: *Found. Comput. Math.* 6.1 (2006), pp. 59–79. ISSN: 1615-3375; 1615-3383/e. DOI: 10.1007/s10208-004-0136-z.

[San11] Raman Sanyal. “On the derivative cones of polyhedral cones”. In: (May 15, 2011). arXiv: 1105.2924v2 [math.OC].

[Tut84] W.T. Tutte. *Graph Theory (Encyclopedia of Mathematics and its Applications).* Addison-Wesley, 1984. ISBN: 0-201-13520-5.

[Zin08] Yuriy Zinchenko. “On hyperbolicity cones associated with elementary symmetric polynomials.” English. In: *Optim. Lett.* 2.3 (2008), pp. 389–402. ISSN: 1862-4472; 1862-4480/e. DOI: 10.1007/s11590-007-0067-0.